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Upgrading Investment Regulations in Second Pillar Pension Systems

A Proposal for Colombia

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Abstract

The passivity of the demand for pension products is one of the striking features of mandatory pension systems. Consequently, the provision of multiple investment alternatives to households (*multifund* schemes) does not ensure that contributions are invested efficiently. In addition, despite the theoretical findings that short term return maximization is not conductive to long-term return maximization, the regulatory framework of pension fund management companies puts excessive emphasis on short-term maximization. Therefore, it is not obvious that typical regulatory framework of pension funds is conductive to optimal pensions. By establishing a set of default options on investment portfolios, this paper proposes a mechanism to align the incentives of the pension fund management companies with the long-term objectives of the contributors. The paper provides a methodology, which is subsequently applied to Colombia.

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Upgrading Investment Regulations in Second Pillar Pension Systems: A Proposal for Colombia

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TABLE OF CONTENTS_

I. Introduction

2. The principal objective of a contributory pension system is to ensure that contributors receive adequate replacement rates at retirement.[2] While in defined benefit systems (DB) the liability for the provision of pensions rests with the program sponsor, which is typically the state or an insurance company; in mandatory defined contribution systems (DC) the adequacy of the expected pension to a great extent falls on the selection of investments made by the contributor.

3. Additionally, while supervisory schemes in DB models put emphasis on ensuring that pension funds have resources to pay the pensions promised, the supervisory focus for DC schemes is on ensuring that the pension funds act within the parameters established by regulation. From this perspective, investment regulation plays a significant role in the future of the contributors in DC funds, and therefore, regulation should aim at pension funds following consistent investment strategies for contributors to achieve adequate replacement rates in the future.

4. Recent literature has shown that life cycle investment strategies are the most efficient strategies from the long term perspective. Properly built life cycle strategies maximize the welfare of individuals, by way of focusing on the long term objectives of the pension funds , for example, Campbell and Viceira (2002), Blake et al. (2008), Rudolph and others (2010).

5. The presumption is that the provision of investment options to individuals does not free governments from the responsibility of providing reasonable pensions, either publicly or privately managed, funded or unfunded. If individuals perceive that their pensions are below their expectations, the demand for change will surely arise. In this vein, the high proportion of government bonds in the pension portfolios in Colombia — as well as in most of the other countries that have reformed their pension system— seems to indicate the need for the asset allocation to be improved in order to ensure that individuals receive pensions that are aligned with their expectations.

6. For most individuals, the amount of future pensions is the only valid tool to measure the performance of a pension fund. Recent literature regarding consumer behavior and financial literacy highlights the inability of average individuals to make investment decisions that are related to long-term horizons; Benartzi and Thaler (2007). In this sense, it is not surprising that the average contributor feels incapable of making portfolio investment decisions, and ends up relying on simple rules or following advice that most likely has little to do with the criterion of optimality. For example, if more than half of individuals adopt poor decisions regarding their pension fund investments, and future pensions fall below expectations, it is likely that this will translate into a political problem for future governments. The developments in the financial literature on this specific subject are incipient, and unlikely to provide practical advice for at least two decades.

[2] The replacement rate is defined as the amount of the pension divided by the final salary.

7. For this reason, mandatory funded pension systems should be able to offer investment strategies that act as a "smart" default option for individuals that are not able to make informed decisions. While people should be free to choose the best investment portfolio that appears within the available options, it is in the best interest of the government to ensure that the funds of those who do not exercise that option are invested in strategies that maximize their expected future pension subject to a predefined set of risks. The development of optimal default options imposes a series of challenges on the investment regulation. Although this approach has not been adopted by other countries with mandatory open pension systems, is common in more sophisticated voluntary pension systems and in public funds.[3]

8. The most practical way of implementing these default options is to allocate individuals who have not made an explicit selection of pension fund by age (and other attributes as appropriate) to a determined pension fund (or a combination of them). The pension fund investment policy should follow the *benchmark* portfolio closely. The *benchmark* portfolio should be designed to maximize the expected long-term returns of individuals. It is proposed that the strategic asset allocation be determined exogenously by a group of "wise persons."

9. The organization of the document is as follows: Section II explains the conceptual motivation of investment regulation; Section III elaborates on this; beginning with the perspective of a benevolent planner, then analyzes the considerations that should guide the investments for a long-term investor; Section IV addresses the causes of and solutions to a possible misalignment of incentives between fund managers and contributors that occurs in pension systems such as the Colombian system; Section V discusses the design of the structural limits of the Colombian pension system; Section VI presents a portfolio optimization model for long-term investors; Section VII analyzes the relative considerations of the benchmark portfolio; and Section VIII concludes with a discussion regarding minimum returns. A detailed explanation of the more technical aspects of the model presented in Section VI is available in the appendices at the end of the document.

II. Fundamentals of investment regulation

10. Due to a number of market imperfections, investment regulation plays an important role guiding pension fund managers towards the investments that maximize the future pensions of individuals. In particular, this paper identifies three areas that justify the existence of regulation: (i) passivity of the demand for pension services; (ii) market structure; and (iii) short-term horizon of pension fund managers.

2.1 <u>Passivity of demand for pension services</u>

11. In a world of rational contributors and multiple investment opportunities, the role of investment regulation is relatively limited. Based on strategic asset allocation, contributors choose a pension fund that best fits their risk profile, taking into consideration a number of factors relevant to them, including age, human capital, the existence of other sources of retirement income, the expected density of contributions

[3] In 2010, Sweden adopted a life cycle framework for its default portfolio.

and risk aversion; de Palma and Prigent (2008, 2009). In addition, individuals change pension funds based on performance (not short term returns), thus the pension fund managers with superior performance tend to grow in comparison with those which underperform. Thus, the market is regulated and portfolio allocations move toward optimal allocations. This (surrealistic) framework assumes that pension contributors know their optimal investment strategy and that pension fund managers then only have to offer these investment strategies so that the contributors may then select the investment options.

12. Empirical evidence, however, has made it clear that these assumptions are not valid, and therefore it becomes necessary to design pension systems based on more realistic assumptions. This will require assuming that contributors do not have a solid foundation to select a pension fund; the investment options are limited; and pension fund managers have strong incentives to maximize short-term returns.

13. In mandatory pension fund plans, people typically choose pension funds strongly encouraged by the sales force. According to Marinovic and Valdes (2005), the main variable explaining the choice of a pension fund management company is a visit by a sales agent. Berstein and Cabrita (2007) also corroborate this finding, but they claim that returns, coupled with a visit from a sales agent, offer a strong fuller explanation. This means that short-term returns are a persuasive rationale to switch to a different pension fund management company. The work of Calderón et al. (2008), using data from Mexico, shows that people generally do not switch to their optimal alternatives.

14. The selection of a pension fund based on its attributes is not necessarily the most appropriate action. Although the cited literature finds clear evidence of price inelasticity, there is confusion by what is meant by the price for members. In particular, these studies take short-term past returns from pension funds to explain the decisions of individuals. However, theory indicates that these variables should not be the principal explanations for portfolio selection, since it is already well-known that past performance is not a good predictor of future returns, and that returns, by themselves, are also not good indicators of portfolio quality. Fund contributors should be more interested in understanding the risk and return profile —something that will allow them to obtain better pensions in the future.

15. Empirical studies suggest that the selection of an optimal portfolio that maximizes their future pension is a decision too complex for an average individual to make. Economic behavior literature shows that people have a limited capacity to understand phenomena associated with pension funds and the level of financial literacy of the average individual, even in the case of the United States, United Kingdom among other developed economies, is too low to be able to make a proper portfolio selection; Benartzi and Thaler (2007). Campbell (2006), in turn, shows that people with lower incomes and less education are more prone to suffer losses as a result of poor investment decisions.

16. Since low financial literacy may impose a heavy tax on lower income people, it is imperative for public policy design of pension schemes to consider this factor. The compulsory nature of the pension funds does not imply that individuals understand the products that are being offered. Therefore, default strategies should be offered to individuals in order to ensure that adequate pensions are paid in the future. Experience from countries with compulsory savings systems and with default options, like Sweden

and Russia, suggest that effective demand from individuals for pension services is relatively small. In these two countries less than 10 percent of the contributors actively select their portfolios; the rest are allocated into the default option. In both, Sweden and Russia, the default option is offered by state-owned entities.[4] Recently, Sweden introduced a life cycle framework into the default portfolio in order to encourage better pensions for their citizens. This portfolio invests 150 percent of its equity value for individuals entering the workforce. [5]

2.2 Market structure

17. Still assuming that contributors are completely rational, the theoretical model assumes that all desired portfolios are available to them. While funds should comply with the personal characteristics of each individual, in practice, the number of portfolios offered is relatively limited and contributors encounter a limited number of investment options.

18. As a way to reduce inequalities within the same cohort, in particular those generated by unscrupulous managers of pension funds, or uninformed contributors, pension fund regulations typically impose short-term performance measurements that reduce the risk of deviations with respect to the average. The minimum guaranteed return is an example of such a measure. It is well-known that this type of performance tool generates a herding effect among pension fund managers (all managers move in the same direction), and therefore their investment portfolios become very similar.

19. With a relatively limited range of investment alternatives, even in a world of extreme rationality, it would not be possible for fund contributors to choose their optimal portfolios. In countries where fund managers may offer only one type of pension fund, contributors face nearly identical investment alternatives and, therefore, there is no major difference among the pension funds selected. In countries with multi-funds, like Chile, Peru, Hungary, and Estonia, the available investment alternatives essentially coincide with the maximum number of alternatives available in accordance with the law, ranging between three to five funds. Funds within the same risk category tend to be very similar. For example, portfolio funds classified as “C” within the Chilean model tend to appear very similar to each other.

20. The experiences of countries that have opened up their investments alternatives for pension funds to any type of risk have not been entirely successful. In a world with limited rationality coupled with fund contributors with little financial knowledge, it is very dangerous to open up the range of investments in accordance with the wishes of managers. In Lithuania, for example, pension fund managers are allowed to offer as many pension funds as they wish. Regulatory requirements are few and are more related to transparency and minimum diversification. The case of Lithuania demonstrates that comparisons among funds become extremely complex and that contributors are not capable of making an appropriate selection. This has led, for example, to some contributors unknowingly choosing high risk funds that were not advertised as such and

[4] Perhaps the effective demand is even smaller. In the case of Russia, approximately 10 percent of taxpayers contribute to work-related pension systems, which from the standpoint of the employee; it effectively is the default option since it is tied to other labor benefits.

[5] This is to say that the fund borrows in order to invest in equities.

suffering major losses during the crisis. It has also created a lack of accountability of pension fund managers. For example, some funds converted their equity positions into fixed income in the middle of the crisis, which resulted in a crystallization of the losses.

2.3 The short-term focus of pension fund managers

21. Designing optimal portfolios for pension fund contributors is not in reality the objective of pension fund management companies. The business of pension fund management companies is simply a business of fees. Pension fund management companies manage people's funds in accordance with parameters established by law, and in exchange receive a fee that allows them to generate income for their shareholders.[6] Since in some cases optimal portfolio design requires taking high short term volatility, pension fund managers may not have the incentive to do so.

22. Laws typically do not impose on managers the obligation of seeking a portfolio that maximizes the welfare of individuals, but they assume —incorrectly— that competition will achieve that objective. In particular, recent studies have shown that the model of managers with competing portfolios tends to skew asset allocations towards short-term portfolios (Basak and Makarov (2009); Castañeda and Rudolph (2010)). The rationality is primarily explained by a manager's motivation to be placed at the top of short-term returns rankings for the pension funds (for example, prevailing compensation schemes), and this objective is achieved with short-term allocations. Consequently, the pension funds end up being managed by criteria similar to those of short- and medium-term mutual funds.

23. Samuelson (1969) and Merton (1969) demonstrate that portfolio selection is independent of the investment horizon only if the following conditions are present:

a. Investors have a utility function with constant risk aversion and an intertemporal elasticity substitution equal to one;
b. Asset returns are independent and identically distributed (iid);
c. Future capital depends on investment returns and not on human capital.

24. The idea that it is optimal to manage pension fund portfolios simply guided by short-term criteria is dismissed due to the strength of these requirements, as the real rates of return are not constant over time, and in the case of pensions, income is primarily from labor and human capital.

25. Offering multiple investment alternatives to contributors is insufficient to ensure good pensions in the future. The most important advancement of the multi portfolio schemes systems compared with systems where pension fund management companies offer a single fund systems is that the former allow portfolio compositions with different exposures to equities. While this allows contributors to eventually capture the risk/reward that equity offers in the long-term, it does not offer a clear solution indication with respect to a number of the risks of these portfolios. According to Blake et al. (1999), Ibbotson and Kaplan (2000), and Iglesias and Walker (2010), strategic asset allocation explains more than 90 percent of pension fund long term returns. Allowing the market to determine strategic asset allocation can lead to sub-optimal equilibrium, especially because it does not take into consideration reinvestment risks,

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[6] There is much debate regarding if the commissions charged for pension funds are reasonable or not, however that discussion exceeds the scope of this document.

inflation risks, the risk reward, or the mean reversion as does the composition of a longer-term portfolio.

III. The benevolent planner and long-term portfolios

26. Since competitive markets are not bringing pension portfolios into the long term equilibrium, it is useful to think on how the benevolent planner would solve the long term equilibrium, specifically in dealing with market risks; concentration risk; liquidity risk; and exchange rate risk.

3.1 <u>Market risks</u>

27. While short-term volatility can be reduced by investing in short-term bond instruments, these investments generate long term risk for the future pensions. While market competition tends to move the equilibrium towards fixed income with short-term durations, theory indicates that for investors with long-term investment horizons it would be optimal to invest in longer-term fixed income instruments; Wachter (2003), Detemple and Rindisbacher (2010).

28. While a risk-free asset in short-term portfolios is a short-term Treasury bill note, a risk-free asset for a pension fund is a government long-term inflation-indexed bond. Nevertheless, there is little incentive for pension funds to invest in long-term fixed income instruments in competitive frameworks as the evaluation of portfolio managers is measured only in terms of short-term returns.

29. Since the planning horizon for a pension fund is understood to be long-term (based on contributor's retirement age), it is optimal for pension funds to reduce reinvestment risk by maintaining long-term government bonds in the portfolio. Investment in these instruments can mitigate the risk of steep drops in real interest rates at the time of reinvesting resources.

30. Pension funds would benefit from more proactive investment in long-term inflation indexed bonds. For example, as a consequence of Colombian economic development, the real rates will probably show a marked decline over the next twenty years, which should be especially attractive for pension funds with long-term planning horizons to lock these relatively high long-term interest rates and transfer those earnings to contributors. It is well known that the purchasing power of money changes significantly over time and it is essential that funds have instruments which at least maintain purchasing power in the long-term. The presence of risk free assets that can hedge inflation risks can help to complete the capital market. The supply of long-term inflation indexed government bonds allows for mitigating reinvestment and inflation risks.[7]

31. Since the objective of a "conservative" fund should be to immunize the pension before an abrupt change in market conditions, assets in this portfolio should correspond to the underlying asset portfolio of a life insurance company that sells annuities at that

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[7] In the case of Chile, the corporate bond, mortgage and infrastructure markets are developed in real terms (indexed to inflation), which allowed investors, including pension funds, to obtain higher returns due to increased credit risk in long-term instruments.

time. Thus, a sudden rise in long-term interest rates, accompanied by a fall in the value of the pension fund is fully hedged by the possibility of buying an annuity with larger annuity payments, and consequently, the value of the annuity to be purchased remains intact. Based on the information available for Colombia, which is similar to other emerging economies, the duration of the fixed income portfolio seems relatively short compared with one that can be extracted from a theoretical model.

3.2 Risk premium and international diversification

31. Empirical evidence shows that it is possible to capture a risk premium through investment in well diversified portfolios of equities. The classic works of Fama and French (1988, 2002) found evidence that a risk premium exists, but it is only possible to capture it in the long-term. More recent estimations [Dimson et al. (2006)] suggest that the risk premium should be about 4.5 percent compared with short-term instruments.

32. Pension funds can capture risk premium by investing in equity instruments in a "consistent" manner. The equity strategy of pension funds should be one of the best selected parameters within the portfolio strategy, and therefore, it has to be consistent over time. Pension funds should not alter their equity exposure before there are changes in the condition of the market. It is surprising that in many countries the exposure to equity in pension funds has had substantial fluctuations since 2007, motivated not only because of the fall in the equity prices, but also by managerial decisions to reduce the equity exposure in the middle of the crisis.

33. The optimal equity investment strategy is that of an internationally diversified portfolio. In countries with underdeveloped capital markets, pension funds typically begin investing in equities in local markets and quickly become the most important institutional investor in the country. While this supports local market development, price behavior becomes endogenous to pension fund investment decisions. After some time, and unless other institutional investors appear on the market, pension funds become trapped with those securities and unable to sell them without affecting their price. In this narrow equilibrium, the only strategy that supports the value of the pension fund is to buy more of the same assets, which creates the condition for a market bubble.

34. Pension funds can feed asset price bubbles, which are not easy to escape. For example, in the second half of the 1990s, Chilean pension funds accumulated a significant proportion of the equity float available on the market. Given the investment restrictions, they had no choice but to continue investing in these local instruments. The situation was "saved" by the arrival of Spanish groups which took control of these companies, the majority belonging to the services sector (electricity and telephone communications). Pension funds in other countries, including Bulgaria in 2007, have helped to feed these domestic asset price bubbles, without a clear exit strategy.

35. Diversification into international markets should happen during the early developmental stages of local markets. However, if left to the market, international diversification is likely emerge too late in the process, probably once the domestic prices are too high and pension fund are heavily invested in domestic equities. While the returns on asset prices in local markets are higher than those of international ones, pension funds have little incentive to invest abroad. This is a relatively short-sighted vision in reality, because while pension funds are what feed domestic prices, it

constitutes an extremely fragile balance. Consequently, pension funds can access the risk reward as long as it happens through the international diversification of their equity portfolios.

3.3 Liquidity risk

36. In pension systems where competition determines reference portfolios (e.g. average return of the industry), managers have the incentive to use liquid assets in order to facilitate trading. However, from a long-term perspective, this strategy may end up being sub-optimal since the resulting cost is imposed on beneficiaries, who should be more interested in achieving a return on assets over the long-term. Since more than 90 percent of pension fund returns are explained by strategic allocation, focusing pension fund investments strategies on short-term trading is counterproductive.

37. Since they are long-term investors by nature, pension funds should be interested in capturing the liquidity premium by investing part of the portfolio in low liquidity instruments. While this is a strategy pursued by some pension funds in developed countries (for example, TIAA CREF and ATP), in pension fund models, such as those in Chile, Peru, and Colombia, illiquid instruments become unattractive.

38. While there is value in investing in illiquid instruments there are also risks of illiquid assets that need to be appropriately measured. Since valuation is a major concern in illiquid markets, investment regulations tend to be reluctant to accept these instruments. Aside from the valuation problems, and since disclosure of illiquid instruments is usually lower compared to public companies, investments with related parties becomes an issue of concern. Therefore, regulation should be careful in allowing these types of investments, and in particular avoiding transactions with related parties, or where there is interest on behalf of the controller of the fund manager. Additionally, the feasibility of investing in instruments of low liquidity should be supported by the institutional framework that ensures fair valuation schemes.

3.4 Exchange rate risks

39. An important decision for investors who invest in foreign currency assets is to decide how much of the exchange exposure to cover in the portfolio. While the majority of institutional investors tend to cover the entire exchange rate exposure, this practice is only optimal if the foreign equity returns are not correlated with the foreign currency returns.

40. Campbell et al. (2010) found that reserve currencies (U.S. Dollar, Euro, and Swiss Franc) tend to be negatively correlated with the returns in global capital markets: these hard currencies tend to appreciate when global markets fall and depreciate when global capital markets increase. This indicates that investors seeking to minimize currency risk in their portfolios should not cover their currency exposure.

41. Chile's experience helps to illustrate this problem. Prior to the crisis, and purely from a short-term perspective, pension funds had an incentive to be covered by the dollar because if markets were rising, the Chilean peso had a tendency to appreciate. If they had not covered themselves, a portfolio in pesos would have had a lower return the following month than that of an identical portfolio but covered by exchange exposure. During the crisis, the value of international assets fell nearly 40 percent, but the peso

depreciated against the dollar to a slightly lesser extent. While the majority of funds were hedged against losses during that period, if they had taken uncovered positions against the dollar the losses would have been minimal.

42. While naked investment operations in foreign securities (unhedged currency exposures) by the pension funds serve as a natural currency hedge in export economies, with increasingly severe problems of currency appreciation, the use of currency hedges by pension funds neutralizes this effect and it can even create the opposite effect. Colombia, like other developing countries, is exposed to short-term capital flows and massive revenues from the export sector that tend to generate sustained appreciation trends in the peso. Since pension funds are dominant market players in small capital markets like in Colombia, pension fund currency operations have an effect on the exchange rate. While investment abroad by pension funds leads to a liquidation of pesos and a purchase of dollars in the international market, the use of hedges neutralizes this effect since the banks that grant coverage must liquidate dollars in the local market.[8] Therefore, from the macroeconomic point of view it makes sense for pension funds to keep their exposures in hard currencies unhedged.[9]

43. From this section it can be concluded that in the absence of benchmarks that may guide investment decisions into long-term results, pension funds will behave as short-term mutual funds.

IV. Aligning the interests of pension fund managers with those of the contributors

44. The backbone of the concept of fiduciary responsibility is the capacity of the courts to provide meaningful interpretations of this concept. In countries with *common law*, like England and the United States, the concept of fiduciary responsibility is meaningful and powerful concept, basically because both supervisors and courts can enforce that duty. In countries with a civil code instead, the concept of fiduciary responsibility is not tangible and it is very difficult for both supervisors and judicial courts to enforce these duties without a law that describes exactly what is meant by the concept. Additionally, the difference between cases in England and the United States and those in countries with civil codes like Colombia is that courts are not prepared to debate these issues. Beyond the arguments that can be presented in court, judges tend to decide according to criteria that reflect purely and exclusively the letter of the Law. To illustrate this point, consider a pension fund in Colombia that invests half of its assets in equities in medium-sized enterprises in the Caribbean. Even an unsophisticated understanding of asset management would point to this as being a high risk strategy and not a rational investment (irresponsible from a fiduciary perspective) for a pension fund. If the companies are destroyed and the assets become worthless and action is taken against the manager, a common law court is likely to look at the actions of the manager in making the investment against what is in the best interests of the contributors. By contrast in a system based on the civil code, a court is likely to be interested in whether

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8 In the Chilean case, Cowan et al. (2007) find a significant effect between pension fund investments abroad with exchange rate depreciation.

9 Unless new evidence becomes available, currency exposures to emerging market currencies should be hedged against a hard currency (e.g. Dollar, Euro).

the investments were made within the limits permitted under the regulations. In countries where the concept of fiduciary responsibility has not been well developed, investment regulations should remain relatively prescriptive.

45. Although it is necessary to have good investment regulations, they might not be sufficient to ensure that funds are invested in a manner consistent with the long-term objectives of contributors. Investment regulations normally prescribe those asset classes in which a pension fund can invest and are silent about the construction of an optimal portfolio. As markets develop, the latitude that regulation grants is so wide that pension funds may move into sub-optimal portfolios, undetected.

46. In most emerging countries that have created 2nd pillars, the structure of investment limits has not followed portfolio risk logic. In countries that are starting their pension reforms, typically with tiny capital markets and due to the limited availability of instruments (deposits, securities, and short-term Treasury bills), the investment regulation for pension funds is relatively simple. When markets become more sophisticated and begin developing other types of instruments, including mutual funds, investment fund shares, derivatives, securitized instruments, etc. it becomes more complex to determine which investment limits are applicable to each instrument.

47. In practice, investment regulations have tended to validate the presence of new instruments in the market and have allowed pension funds to keep investing "little by little" in more sophisticated instruments, under the assumption that the investments are secure as long as there are not dramatic price corrections. Based on the experience of Chile (before the legislative changes in 2008), it is possible to argue that the latitude granted for these instruments is far from having any economic or financial rationality of risk, and the investment liberalization process has followed a gradual approach of the validation of availability of instruments.

4.1 International experience

48. Due to a multiplicity of factors that explain an optimal portfolio, it is expected that investment regulation are different among countries. The most important are the following:

a. Existence of other sources of retirement income

49. While in some countries like Chile and Colombia, all pension income comes from a mandatory capitalization system, other countries like Poland and Lithuania count on a first pillar system that allows them to ensure an income independent of what the individual capitalization system yields.[10] Thus, investment regulation in countries with a first pillar system should be relatively more flexible than in countries where the pensions depend exclusively on the second pillar.

b. Supervisory approach

50. In supervisory approaches that are compliance based, the responsibility of managers is to comply with the limits established in the law and regulations. However, in risk based supervision frameworks, managers have to prove that they have adequate

[10] In 2008 Chile introduced a solidarity pillar that mitigates the risk of poverty for pensioners.

capacity to manage risks before being able to invest in a specific asset class. Thus, investment regulations in systems with compliance based supervision should be much stricter than those where the approach is risk based.

c. The degree of development of the domestic capital market

51. In countries with relatively developed financial markets, the domestic capital market is capable of absorbing pension fund investments without a significant effect on asset prices, while in countries with small capital markets, pension fund investments can have a significant impact on domestic asset prices and therefore create speculative spirals. The experience in Latin America and Central Europe shows that in countries with small capital markets, the size of the pension funds is likely to outgrow the availability of instruments in the capital market in the first decade after the reform. Thus, in order to avoid asset prices bubble, the investment regulation of pension funds in small capital markets should be stricter than in countries with deeper markets.

d. Contribution rate

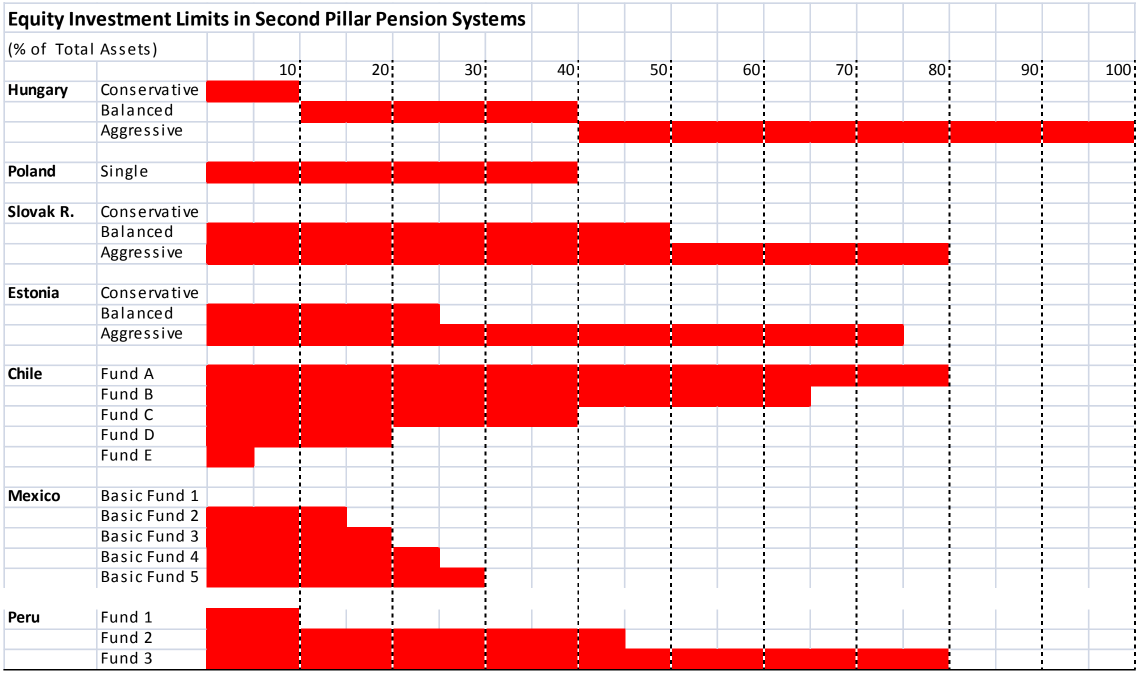
52. All other things equal and assuming that individuals are trying to reach a certain replacement rate, investment regulation in countries with higher contribution rates have less pressure on the risk of the portfolio compared with those with lower contribution rates. In incipient markets, regulation should also be mindful of the stability of the demand for public bonds. Pension funds play an important role as buy and holders of government instruments.

4.2 Comparative analysis

53. As described in the previous section, investment regulations across countries may target different risk profiles. As seen in Figure 1, risks profiles embedded in the investment limits of different countries tend to be diverse. In the most aggressive fund in Mexico, an investment of up to 30 percent in equity instruments is permitted, while the riskiest fund in Hungary was required to make an investment of at least 40 percent.[11]

[11] Funded pension funds were nationalized in 2011

Figure 1: Equity Investment Limits in Countries with Second Pillars, 2009



Source: World Bank

54. One element that requires consideration is that the majority of the investment regulations have moved towards limits without minimum thresholds. Until its recent nationalization, Hungary was probably the only system that required minimum investment limits. In Chile there were thresholds for investments in equities in the past, but these were replaced by an approach which requires the most conservative funds always to have less equity than the most aggressive ones. This change is due to the fact that in the previous legislation there was the possibility that a risky fund might have less equity than a conservative fund. Slovakia also requires more conservative funds to have less equity than riskier funds.

55. By not imposing limits on equities, the market is left to decide the exposure to that asset class, which in no case ensures optimal allocation. For example, in the case of Slovakia, balanced and aggressive funds are allowed to be invested in equities up to 50 and 80 percent, respectively. However, in 2006 the actual allocation amounts were only 6 and 8 percent for the balanced and aggressive funds respectively. In 2009 the actual equity participation in the portfolios was further reduced to approximately 1 percent of the total assets. Since then, differences among the three types of funds are negligible.

56. The imposition of minimum thresholds for certain asset classes requires a clear understanding of depth of the market, the pension system risks, from the regulatory authority that many countries are not willing to assume. If the regulatory authority does not have a clear expectation of pension system risks, it is even more unlikely that the contributors may have one. In the absence of a minimum investment thresholds for certain asset classes, portfolio allocations that might emerge from the mere imposition of a ceiling can be very varied. In addition, this can lead to erratic allocations in the behavior of equity allocations, which might not be related to the strategic positions, and which are very difficult to justify not only from an optimality point of view. Perhaps the only advantage in avoiding the imposition of lower limits is to give the feeling of less intervention.

V. Structure of investment limits in the Colombian pension system

5.1 <u>Limits with related parties and conflicts of interest</u>

57. Financial infrastructure and transparency are necessary conditions in order to prevent misappropriation of fund assets. In this sense, the separation of the role of the pension fund manager from that of the fund has played an important role in the security of the contributions. Additionally, the presence of independent custodians has been essential to protect the ownership of the funds. High levels of transparency and the existence of a specialized supervisory authority have also played important roles in ensuring that managers are responsible for managing third-party resources.

58. However, transparency is not a sufficient condition for aligning the incentives of pension funds with those of the pension fund management company. Limits with related parties are a controversial issue which goes to the heart of the alignment of interests of the manager and the contributor. To the extent that managers are allowed to invest part of the fund in companies related to the controller, they are serving shareholder interests and not necessarily contributor interests. In developing countries with small capital markets, it is common to find that the controllers of pension fund managers are also controllers of companies in the real sector of the country, and surprisingly investment regulation tends to validate these conflicts. With the strong competition for market share from international financial groups, it is hard to justify regulations that favor the presence of local groups in the ownership of pension fund management companies and consequently face serious conflicts of interests when making their investment decisions.

59. The relationship between pension fund managers and economic groups belonging to the same controller becomes a sensitive factor to the extent that the controller has the incentives to use pension funds to finance its own operations. In countries where legislation regarding related party transactions is relatively formal and does not extend to indirect ownership relationships, this phenomenon happens more frequently. It generates, therefore, an unfair competition with other participants and a wider disconnection with the objectives of the contributors, as investment decisions are more guided by the interests of the controller than the ones from the contributors.

60. In the case of financial groups, it is also risky when pension funds begin to operate in conjunction with a bank or any another financial entity of the controller. For example, it is common to encounter cases in which group brokerage launches a bond and at the same time one of the largest buyers of the bond is the pension fund management company within the group using the pension fund. Another common example is when the bank in the financial group manages the short-term assets of the pension fund. This situation typically happens when the bank CFO, or some of his or her associates, is also member of the board of the pension fund management company. Although international diversification in instruments where the financial group has some interest has lower risk through greater diversification, it is risky from the perspective that pension fund investment can be used for financing risky operations of the financial group.

61. Detecting transactions with related parties and initiating investigations in the local markets become more complex once pension funds invest internationally. For example, in 2003 *Telefonica Chile*, a publicly traded company, controlled by Telefonica (Spain) decided to split the mobile business from the rest of the telecom business, and to buying the minority shareholders out of the mobile business. The transaction requested to be voted at the shareholders assembly, and the vote of the majority of pension funds was essential for its approval. While the pension funds did not have a common view about the price of the transaction, the vote of *Provida*, the largest pension fund in Chile, was necessary for completing the transaction. BBVA was the controller of *Provida*, and also (at that time) the main shareholder of *Telefonica-Spain*. With the support of Provida, the (controversial) transaction was completed. *BBVA* (Spain) and *Telefonica* (Spain) simultaneously decided to initiate a joint venture for exploring the possibilities of mobile banking in some Latin American countries. This specific transaction was decided by the board of directors (mostly represented by *BBVA* executives) of *Provida* and not by the *Provida*'s investment manager, as all other transactions. Despite the controversy in the specialized press, the Chilean pension supervisor did nothing about it. The cost of initiating such investigation would have costly for the supervisor, and the probability of getting something concrete was minimal, as most of the decisions were taken outside the Chilean frontiers. Despite the "good story," it would have been difficult to prove any wrongdoing by *Provida* and its shareholders. Finally, Chilean courts would have been unable to get into the substance of the transaction.

62. Two solutions might be proposed for addressing this problem, but both involve reducing the investment limit of related parties to zero. The first alternative consists of raising the requirements to be a pension fund manager, such as requiring managers to renew their license under higher operation standards. The second option is to require that the majority of directors of pension fund managers be independent directors, not only from the perspective of holding other positions within the group but also from having any business relationship with the group or any personal relationship with the main shareholders.

63. The introduction of contributors in the decision making of pension fund has not worked as a mechanism for improving performance of pension funds. The experience of Hungary demonstrates that it is neither realistic nor efficient to try to incorporate contributors into the investment decision-making process. Evidence suggests that the incorporation of contributors into management decisions of the pension funds has resulted in practice, in formal ceremonies that contributed little to nothing to improving the pension system. As an anecdote, pension funds have had difficulty in filling the participation quorums required by law in the assembly, and typically only employees of the management company are those who actually show up at such meetings. Assuming that contributors are interested in what is happening with pension funds and that they have the capacity to provide meaningful inputs to each pension fund management company are not realistic assumptions. Finally, organizing contributors so that they might become more involved in the decisions of the pension fund can be very expensive, and with an unclear outcome.

5.2 Structural limits

64. For an economy like Colombia, the regulation should basically establish four investment limits. Other existing limits may be substituted by improvement in the risk management requirements of the pension fund management company. While in a

compliance based supervision scheme, it is necessary to establish issuer limits, issuance limits, and concentration limits in order to keep managers from taking unnecessary risks, in a risk based supervision approach, with exception of limits with related parties, the pension fund manager should design strategies to mitigate those risks through their risk management process or the internal investment guidelines of the fund. The convenience of introducing additional limits in Colombia would depend on the speed of progress in the areas of supervision.

a. Equities

65. Equity investments in the life cycle context are very important consideration as the quantum in the portfolio varies according to the contributor's age. Limits on these parameters must be defined as precisely as possible, including maxima and minima without any overlap between collars. These collars or bands (among which should be investment in equities) hopefully need to be designed according to models of long-term optimization.

66. Unless the use of a benchmark portfolio is imposed, it is insufficient to impose only maximum limits in equities, since the market equilibrium can lead to situations of severe underweight in equity exposure, which may result in low levels of future pensions.[12] In a rules-based system, it is necessary to have limits on foreign and local equity, while in monitoring systems based on risk, it is sufficient to have a single limit for equities. By not having limits on local equity exposure, the system runs the risk of organic growth of pension funds overtaking the growth in the market which creates an equity asset price bubble, as described above. In the case of risk-based supervision, risk management models should consider such types of risk.

b. Currencies

67. Where pension funds invest overseas and the securities are denominated in either US dollars or in Euros, these positions should not be hedged. If pension funds invest in overseas securities denominated in other currencies, hedging should be permitted but limited to hedging the position against the US dollar or the Euro. Hedging is a complex issue which needs to be the subject of a comprehensive regulation. Where a futures market exists, hedging should be done in this market; otherwise in the forwards market by installments, for which there is greater liquidity. For example, if pension funds have exposure to the Brazilian *Real*, the coverage of *Reales* to dollars can be done in futures markets.[13]

c. Government bonds

68. Pension funds are expected to hold investments in risk-free assets. Since the maximum government bond exposure in Colombia is an issue stipulated by law, emphasis in the regulation should be focused on the duration of such assets. Regulation may establish a minimum limit for the duration depending on the type of fund. Thus, riskier funds should require a longer duration in government bonds than more conservative ones. Limits in this area should be subject to the availability of instruments

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[12] In the case of Slovakia, which was illustrated in the previous section, in spite of relatively high limits established by law, equity investment is less than 1 percent.

[13] See Viceira (2010), Campbell, Serfaty-de Medeiros,Viceira (2010), and Walker (2008)

on the market, and the government should play a proactive role in supplying government securities that may fit the needs of pension funds.

Box 1: Coverage for Minimum Wage Increases

Since it creates sizable price distortions in the pricing of annuities, the inability to provide hedge against minimum wage risk is one of the major challenges that the Colombian private pension system. Life insurance companies (CSVs) are reluctant to sell annuities to people who receive a pension up to one and half times their minimum wage by way of an annuity because of the risk of triggering the minimum pension at some point in the future. At the same time, annuities for people with slightly higher funds are severely discounted and individuals end up receiving pensions close to minimum wage because the CSVs do not have instruments to cover the risk of this variable increasing.

The lack of instruments to cover the risk of minimum wages increases the poverty level at old age. The lack of this instrument is equivalent to imposing a regressive tax, which can reach up to 50 percent of the value of the asset pension for individuals with low income.. People who expect to receive three or more minimum wages receive fairer pensions because the risk of triggering a minimum pension in low.

This paper proposes a mechanism for the government to internalize the costs generated by increases in minimum wages. Under the current framework, the impact of the fiscal budget of increases in minimum wage is relatively modest,[14] so governments may find it relatively inexpensive to increase minimum wages. In order to align incentives, the government with the rest of the society, the government should pay a cost of raising the minimum wage. This can be achieved via the options market.

The government can address this problem by issuing long-term inflation indexed bonds, but at the same time, have an option attached to these instruments that pays the maximum between inflation and the minimum wage. In this manner, various objectives are achieved simultaneously. First, life insurance companies begin paying fairer pensions and they have an incentive to sell annuities to people with lower incomes. Second, the government begins to take a cautious position with respect to future increases in minimum wage, since there are permanent effects on the budget.[15] Next, it allows a robust and strong demand for long-term inflation indexed government bonds, without the need of segmenting the capital market.[16] Finally, it allows the government to refinance their public debt in a long horizon. It is important to highlight that the provision of this type of instrument requires the development of expertise in the Ministry of Finance in option valuation.

[14] In the majority of the countries, a high quantity of fiscal employees receives remuneration significantly higher than the minimum salary.

[15] An increase of 1 percent of the minimum wage above the value of inflation in the first year, with a 20 year bond generates a difference in value of about 22 percent.

[16] The alternative to issuing bonds indexed to minimum wage is less efficient because it generates an additional segmentation between nominal bonds, inflation-indexed bonds, and bonds indexed to minimum wage.

d. Catastrophic risk

69. Pension funds may want to avoid the risk of losses during a given period through the use of stop loss instruments. Derivatives are a powerful instrument to mitigate these risks. Regulation may provide guidance on the type of instruments that can be used to hedge catastrophic risks.

VI. The long-term portfolio and the optimization process

70. In the context of portfolio management, strategic asset allocation (SAA) represents the materialization of the investment objectives of the individual investor, and is expressed by the allocation of funds available for investment into the different asset classes, according to the stipulations of the management contract.

71. Naturally, the allocation in question should be compatible with restrictions from the management contract (for example, investment limitations in certain, specific assets), and with the prevailing conditions in the market (e.g., expected returns and volatility of assets on the market, etc.). See Maggin et al. (2007).

72. In the specific case of pension funds, the definition of the SAA is a matter of highest importance, as recent evidence has shown that this is responsible for a large percentage of the variability of returns earned by investors, including pension fund managers (Brinson et al. (1986, 1991), Blake et al. (1999), Ibbotson and Kaplan (2000)).[17]

73. The above-mentioned point is especially important when pension funds are operating in countries with a DC pension system, since the SAA definition involves aspects that relate not only to the mandate or the prevailing conditions in the securities markets, but also to those related to the life cycle stage of the contributor. In this respect, financial literature points out that the optimal SAA of a pension fund should consider the life cycle strategy of individuals (Bodie et al. (1992)).

74. Alternative models, such as the “no lose” suggested by Feldstein (2005) and tested by Poterba et al. (2006) for the American economy are difficult to test in economies with incomplete fixed income markets like Colombia. Unlike life cycle models, no lose models are not derived from optimization models and are intensive in data availability, since the amount of equity depends on the historical return of long-term, inflation-indexed bond curve and by equity returns. The lack of an inflation-indexed bond yield curve was also an impediment for testing the convenience of a no lose strategy.[18]

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[17] The studies by Brinson et al. and Blake et al. found that more than 90 percent of the variability in returns obtained by investors over time is explained by the SAA, while Ibbotson and Kaplan found that the SAA explains over 40 percent of the variability of returns among fund managers.

[18] The lack of data leads to completely deterministic predictions.

75. Life cycle models are based on the assumption that the wealth of younger people is primarily determined by human capital (that is, the present value of wages they will receive as a product of their work). Under this assumption, if the individuals profiles are such that their risk-return preferences imply that they wish to maintain a percentage of their total wealth (i.e., human capital plus financial wealth) invested in risky assets (like stocks, or long-term bonds),[19] the optimal SAA will crucially depend on the human capital characteristics of the individual. Although there are differences among human capital profiles that may justify different portfolio allocations, there is also room for grouping individuals with slightly different profiles.

76. For example, if human capital is similar to an investment in bank deposits,[20] then the SAA will be such as to allocate a percentage of $X\% \times (1 + f_t^{HK})$ of financial wealth of risky assets, where f_t^{HK} corresponds to the fraction representing human capital within the financial wealth at the moment t. This result basically explains that by noticing if the individual in question wishes to maintain $X\%$ of their total wealth invested in stocks, and a fraction f_t^{HK} of their financial wealth is indirectly invested in deposits, it therefore requires an increased stock investment of $X\% f_t^{HK}$ in order to achieve the desired combination.[21]

77. Figure 2 illustrates this classic result for the expected trajectory of returns. The figure was constructed based on a simplified version of the model developed in Appendix A (with two assets: equity and fixed income instruments) and takes into account the particularities of the Colombian context: a risk premium of 4.5 percent (consistent with the MSCI World Emerging Markets), a risk aversion coefficient of 2.5 (consistent with a conservative approach[22]), and an individual who at the age of 20 receives a monthly remuneration equivalent to 0.5 minimum wages, which subsequently evolves according to the investment profile documented by Viceira (2010, pg. 223) for Chile, with an average between 1 and 2 minimum wages throughout his/her working life, which is consistent with the reality of the pension system in Colombia.

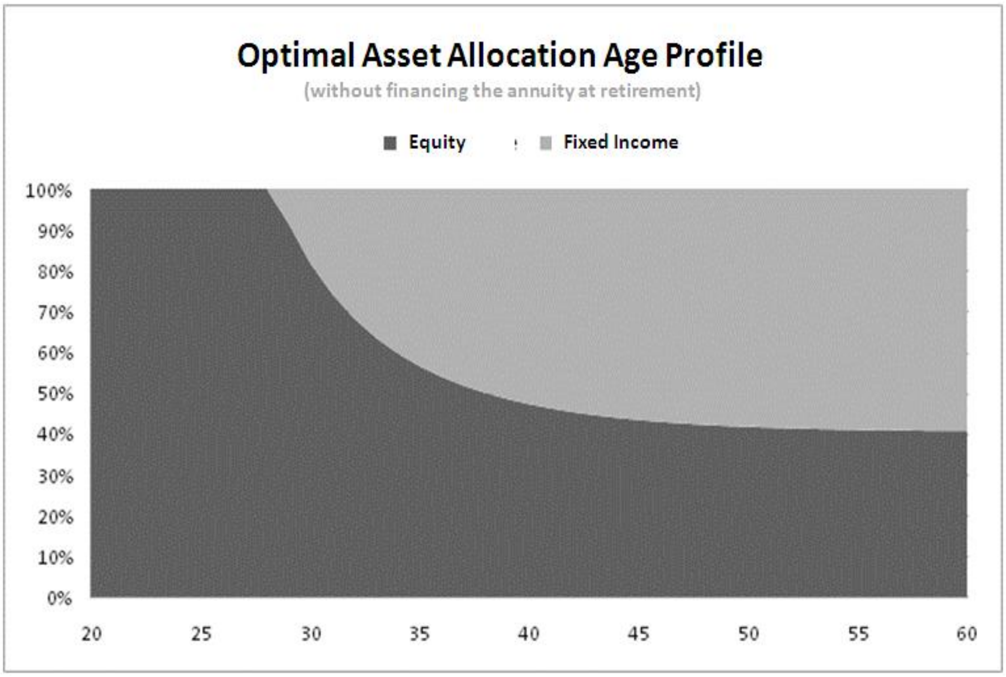
19 The assumptions necessary to justify the indicated characterization can be found in Appendix A, or alternatively, in Merton (1969). In short, what is required is that the individual has preferences (expected utility) with constant, relative risk aversion and a set of investment opportunities (expected return, short-term interest rate and market volatility) that remains constant over time.

20 Assumption that is consistent with the low correlation (close to zero) between wages and stock returns [Cocco et al. (2005, pg. 500), Viceira (2010)], although Campbell (1996) argues otherwise.

21 For more clarity, note the total wealth given by: $W_t^{total} = W_t^F + W_t^{HK}$, from where you have $1 + f_t^{HK} = f_t^{total}$.

22 A coefficient of relative risk aversion of 2.5 implies that the individual will be indifferent between entering a fair bet where they can increase or decrease their wealth by ±*50%* and surely lose 30% of their current wealth.

Figure 2: Optimal portfolio profile



Source: Authors' own calculations.

The profile illustrated in the figure shows that the expected trajectory of equity instruments (and other risky assets) is decreasing throughout the individuals' life cycle, which is explained by the decrease of f_t^{HK} as the individual ages and their human capital depreciates. At the end, when the latter has fully depreciated, investment in risky assets reaches precisely X% of financial wealth (and at these levels equals total wealth) of the individual. Please note that for the parameters employed $X = 40$; that is to say that at the end of the investment horizon the individual wishes to maintain an exposure of 40 percent in stocks. This result assumes, however, that the individual's risk aversion remains constant throughout their lifetime, which is not evident in the case of an individual who is completely dependent on their pension savings. In particular, if the coefficient of risk aversion is duplicated as the individual ages, the inverted fraction invested in risky assets would be reduced by half (which is, $X = 20$ if $R = 5$).

78. This paper makes references to this result as the "base case," and utilizes it as a point of reference to analyze and evaluate relevant factors in a defined contribution pension system that could alter or modify its outcome.

6.1 <u>Analysis based on a parametric model</u>

79. In order to motivate the analysis of principal elements affecting the outcome of the base case**,** this section presents a summary of a portfolio selection model developed in Appendix A.

80. The model is part of the tradition of portfolio selection models from Merton (1969, 1971) and Samuelson (1969), including more recent advancements [Detemple et al. (2003, 2005), Detemple and Rindisbacher (2010)] and is comprised of three dimensions:

- pension system investment objectives
- restrictions surrounding its operation, and
- investment opportunities for pension funds

81. In terms of the objectives of the pension system, the model adopts the premise that these are given by the objective of maximizing the representative individual's welfare (which is directly related to the pension amount that they will receive after retirement), and additionally, by guaranteeing a minimal level of income during the inactive stage of the life cycle.

82. To this end, the algebraic description of the problem considers as argument of the objective function (of the expected utility type) the pension that the individual is capable of financing, instead of a simple terminal value of the individual account at the moment of pension:

$$U(X_T) := \mathbb{E}[e^{-\rho T} u(X_T/P_T - \eta \bar{c}_T)]$$

where $\mathbb{E}[\cdot]$ corresponds to the mathematical expectation, $\rho \geq 0$ is a discount factor, $u(\cdot): \mathbb{R}_+ \to \mathbb{R}$ is a function of instantaneous, increasing, concave utility, that has the amount of the pension possible to finance (X_T/P_T) as the argument, which depends on the accumulated amount in the individual account $(X_T \geq 0)$ and the unitary price of annuities upon retirement $(P_T \geq 0)$. Additionally, the objective function considers the fraction $(\eta \in [0,1])$ of minimum wage at the time of the pension $(\bar{c}_T \geq 0)$, as the minimal pension level.

83. The above-mentioned objective function permits the SAA to consider the need to cover adverse fluctuations in the factors that influence the price of annuity $(P_T \geq 0)$, hereby avoiding the occurrence of a disconnection between the performance of the pension fund and the pension that the individual will receive at the time of retirement. Additionally, this function also considers the fiscal interest as an incentive for the accumulation of sufficient funds for financing, at least, a fraction $(\eta \in [0,1])$ of minimum wage $(\bar{c}_T \geq 0)$, and imposes a penalty in the case this does not happen (since: $u(x) = -K$, for $x < 0$, with $K > 0$).

84. In addition, the considered investment restrictions are taken into account by setting the admissible investment rule as $\pi_{it} \in [\alpha_{it}, \beta_{it}]$ where π_{it} is the percentage of the pension fund invested in the asset class $i \in \{1, \ldots, d\}$, at the moment $t \in [0, T]$, while α_{it} and β_{it} correspond to the lower and higher investment limits, respectively, for that asset class.

85. Finally, investment opportunities available to pension funds were characterized by a set of $d + 1$ asset classes.

6.2 <u>Specific considerations</u>

86. Based on the parametric model description, we discuss five aspects that influence the result of the base case:

a. relationship with 'efficiency' in the sense of mean-variance;
b. effects of human capital characteristics in strategic asset allocation;
c. effects of considering minimum pension objectives;
d. effects of parameter uncertainty; and
e. effects of portfolio restrictions and incomplete markets.

First, it is worth noticing that when an individual's total wealth comes entirely from their financial wealth (that is, that human capital is equal to zero) the base case recommends investing $X\%$ in stocks. However, Merton (1969) shows that in the case with two assets (for example, stocks and fixed income instruments) and constant investment opportunities over time, this percentage simply corresponds to $(R\sigma^2)^{-1}(\mu - r)$, where $R > 0$ is the relative risk aversion coefficient, $\sigma > 0$ is the volatility of the risky asset and $\mu - r > 0$ is the market risk reward (with $\mu > r$ as the expected stock return and r is the interest rate of a bank account), an expression which, in turn, yields an identical solution (interior) to the problem of short-term mean-variance:

$$max_{\pi\in[0,1]}\{r + \pi(\mu - r) - (R/2)(\pi\sigma)^2\},$$

which corresponds to one of the possible specifications of the problem studied by Markowitz.

87. Consequently, the recommendation from the base case (that is "[…] to invest a decreasing amount over time in volatile assets […]") can be understood as an extension of the result of mean-variance efficiency by Markowitz, for the case where the portfolio selection is dynamic and is carried throughout the life cycle of the individual, which considers variations in the financial situation of the individual.

88. Naturally, since both are complementary results, the base case shares the intuition and spirit of the classic result by Markowitz, but also its shortcomings. In particular, the instability of the results when faced with parameter uncertainty (e.g., expected returns, elements of the variance-covariance matrix, etc.); Rachev et al. (2008, pg. 247).

89. Additionally, when the data generating process of asset returns vary over time, it is possible to show that Markowitz's recommendation is modified in favor of a solution that contains time-varying hedging demands, that reflects the ability of long-term investors to anticipate (or cover) the adverse fluctuations in the investment opportunity set; Merton (1971, 1973). This is the case, for example, when the stock returns exhibit mean reversion; Kim and Omberg (1996), Campbell and Viceira (1999, 2001).

90. An important corollary that branches from this last point is that, in the presence of a time-varying investment opportunity set (mean reversion of returns, variable volatility, etc.), the optimal SAA fails to be (static) mean-variance efficient. Hence, evidence suggesting the "inefficiency" (in the mean-variance sense) of long-term pension fund portfolios does not imply the presence of a sub-optimal investment portfolio, but rather the complete opposite, if you look at it from a long-term investment horizon point of view.

91. Second, it is worth recalling that when the contemporary correlation between human capital returns and the stock market is low (assuming from the base case scenario), human capital is comparable to an indirect investment in fixed income instruments, so that the financial investment in assets increases by $X\% f_t^{HK}$ in order to achieve the combination between stocks and the desired fixed income. However, there are reasons that suggest that human capital can be linked to an investment with a significant percentage of equity.

92. One example is the relationship that exists between a sharp fall (rise) of the stock indices that precedes a subsequent increase (decrease) of unemployment in the economy; Stock and Walson (2003). This fact can be understood as a positive correlation (not necessarily contemporary) between stock returns and returns of human capital.

93. Third, another aspect that is relevant for evaluating the optimal SAA has to do with pension system objectives, that is, to finance a minimum consumption during the inactive labor period of an individual (consumption smoothing). The importance of this element has far-reaching effects for determining the optimal SAA and deserves to be discussed in more detail.[23]

94. Specifically, the introduction of a minimum consumption (or income) level makes an individual who wishes to maintain $X\%$ of their total wealth invested in stocks, to effectively invest (in the case that $\sigma^{HK} = 0$) a percentage of:

$$X\% \times \left(1 - f_t^{\bar{c}} + f_t^{HK}\right) + \left(\sigma^{\bar{c}}/\sigma\right) f_t^{\bar{c}}$$

in stocks of their financial wealth, where $f_t^{\bar{c}}$ is the fraction of financial wealth needed to finance the present value in the moment t of consumption that requires financing, while $\sigma^{\bar{c}}$ is the volatility of this last amount. In this case, the quotient $(\sigma^{\bar{c}}/\sigma)$ maintains the sign of correlation between the stocks and minimum consumption level. As a consequence, investment in stocks depends on two factors. First, whether the total wealth of the individual is or is not sufficient to finance the value of consumption or minimum income (that is: $1 + f_t^{HK} \gtreqless f_t^{\bar{C}}$); and second, the sign of correlation between both quantities (that is, $\sigma^{\bar{C}}/\sigma \gtreqless 0$)).

95. To analyze the effects of this case, assume initially that consumption (or the minimum income level) is an amount that is not associated with stock price ($\sigma^{\bar{c}} = 0$). Hence, equity investment will always be positive as long as the (total) wealth of the individual is sufficient to finance the present value of consumption flow or minimum income ($f_t^{\bar{c}} < 1 + f_t^{HK}$). Consequently, the individual will only invest in stocks if they have a surplus of resources, after having met the obligation to finance consumption (or the minimum income level).

96. The intuition of this is that if the representative individual possesses just enough wealth to finance the minimum level of consumption [e.g., ($f_t^{\bar{c}} = 1 + f_t^{HK}$)], the optimal SAA will imply investing 100 percent of its resources in bank deposits, since this is the way to replicate the future value of this quasi-obligation (or liability) when $\sigma^{\bar{c}} = 0$.[24] Now, if instead the future obligation is perfectly correlated with the stock price ($\sigma^{\bar{c}} = \sigma$) including an individual with just enough wealth to finance the minimum consumption flow, they will be able to invest the entirety of their financial wealth in stocks, since this method enables them to cover the obligations they face.

97. In agreement with the aforementioned, the motivations to invest in stocks can be diverse. For example, when the total wealth of an individual is hardly sufficient to

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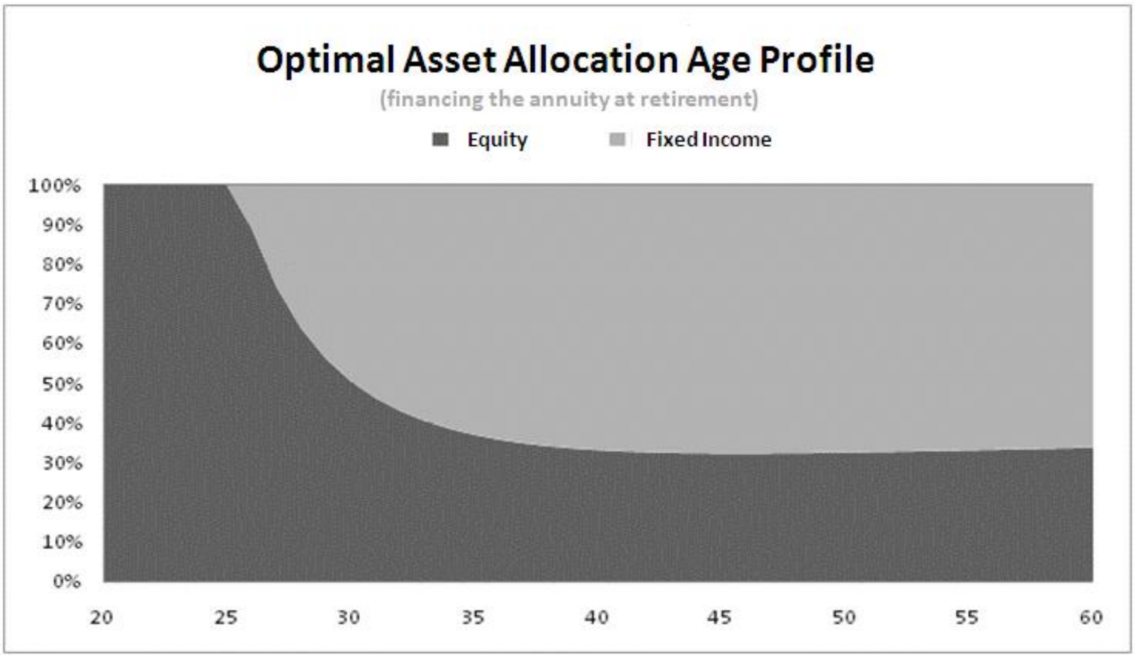
[23] The analysis of this point is based largely on Castañeda and Fajnzylber (2008).
[24] This model excludes the possibility of "gambling for resurrection."

finance the above-mentioned requirement, the investment in shares (if any) will be possible only by the similarity between the behavior of stock prices and the value of the obligation. Meanwhile, when the wealth of an individual is sufficient to cover the obligation, equity investment is motivated by stock risk reward (implicit in $X\%$), which can be increased or decreased depending on whether $\sigma^{\bar{c}} \gtrless 0$.[25]

98. Figure 3 illustrates this situation for the trajectory of expected returns, under the assumption that the obligation at the time of retirement corresponds to an annuity equivalent to the minimum wage.[26] The primary difference with the figure presented in the base case (Figure 2) comes from the reduced availability of resources to privilege the risk reward. Although it is important to note that, again, a constant grade of risk aversion has been maintained. Increased risk aversion at the end of the active stage of the individual would implicate less investment in risky assets.

Figure 3: Optimal portfolio profile for low income workers



Source: authors' own calculations.

99. As shown, the introduction of a minimum level of consumption during old age has strong implications for an optimal SAA. Particularly, the optimal SAA crucially depends on the similarities and differences between financial assets and the implicit liabilities in the pension system (that is, the level of consumption or minimum wage that is being sought after), combined with the financial situation of the representative individual for which the pension fund investments were designed.

100. Another important element refers to the case in which the obligation in question corresponds to a pension or minimum wage for life (or life annuities). In the latter case,

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25 Note that the presence of financial obligations is another reason for which the SAA fails in being efficient in the mean-variance sense. To see this, it suffices to consider the presence of an obligation that can be perfectly covered by investing in a single asset.

26 The figure was constructed using the same parameters as in Figure 2, with $\sigma^{\bar{c}} = 0$.

the obligation takes the form of an annuity whose payment is deferred until the individual retires. Then, since the price of an annuity can be replicated by a portfolio of government long-term bonds [see, for example, Milevsky (2006)], the analysis suggets that stock investment will be determined, on one hand, by the monetary sufficiency to finance the annuity in question, and on the other hand, by the correlation between stock prices and deferred annuities.[27]

101. Fourth, an additional aspect that is relevant to mention is related to the uncertainty of the parameter values surrounding portfolio selection, and must be estimated before the investment decision. In this sense, recent literature has shown the uncertainty related to the ignorance of the true value of the parameters associated with the process of generating asset returns, which results in a decline in investments for those assets whose returns are uncertain; Uppal and Wang (2003). The mechanism in question operates via an increase in the effective risk aversion that the individual exhibits when faced with the aforementioned uncertainty.

102. For purposes of the SAA, these findings suggest that the greater the uncertainty about the parameters governing the dynamics of returns the more conservative the portfolio selection should be. This means, in agreement with the previous discussion, that the SAA should be assigned a lower prevalence towards the motivation associated with taking advantage of the asset risk reward.[28]

103. Finally, the last aspect that affects the recommendations emanating from the base case is given by the quantitative investment restrictions. In particular, there are two types of restrictions: 1) those that arise from regulations, and 2) those that originate from market incompleteness. In the first case, these restrictions are motivated by the desire to limit the range of decisions to fund managers. Usually, these restrictions seek to limit investment in certain asset classes (for example, low liquidity stock shares, high yield bonds, etc.), or alternatively, they can be motivated by the use of some restrictions of volatility of VaR (Value-at-Risk) portfolios. The second type of restriction, on the other hand, comes from the degree of development of financial markets and consists of restricting instruments of investment (or asset classes) that are potentially attractive to pension funds, but that are not available for market investment. This would be the case, for example, of a deferred annuity, an instrument that would be attractive to cover the risk of a price increase in the cost of financing a pension, but that is unavailable due to the inexistence of a sufficiently long returns curve.[29]

104. The relevance of these restrictions comes from the effect they have on the investment in assets that are restriction-free, which have the property of being correlated with those facing restrictions. This is the case, for example, of long-term bonds and stocks. These are assets whose value tends to move in the same direction. Due to this,

[27] Note that Figure 3 was created under the assumption that the rate structure is flat, so the hedging of the changes in the annuity price fluctuations is due to investing in deposits. When the interest rate structure is not flat and the interest rates fluctuate randomly, hedging requires investing in long-term bonds.

[28] Note that as long as equity is a good asset to replicate the annuity price, the recommendation does not necessarily imply to reduce equity to zero.

[29] Conceptually, both restrictions can be accommodated within the parametric model; that is, for example, $\pi_{1t} \leq \overline{\pi}_{1t}$ and $\pi_{2t} = 0$,, where π_{1t} corresponds to the percentage invested in the class of assets subject to quantitative restrictions, and π_{2t} the percentage invested in the asset that is not available for investment.

investing in stocks can be motivated not by an equity risk reward, but by the capacity of this asset class to indirectly hedge pension funds against increases in the cost of financing a pension.

6.3 Conclusions from the model

105. In practical terms, the model is consistent with a profile of decreasing investment in equities throughout the life cycle, mainly due to the gradual depreciation of human capital and the fact that it is comparable, to an extent, to an investment in long-term bonds.

106. The limited availability of reliable long term data for Colombia suggests, however, that the numerical results of the model should be taken with caution on behalf of public policy makers. Fortunately, the theory of portfolio selection has shown progress in cases of high uncertainty with respect to the most important parameters of the model, and suggests that investment regulation should promote more conservative portfolios as a way of mitigating the data shortage.

VII. Reference portfolio

7.1 The benchmark

107. This paper has argued for the need to establish a long-term reference portfolio that serves as a basis for designing the pension fund portfolio strategy. Taking into consideration certain peculiarities of the Colombian case [see, for example, Rudolph et al. (2007)], the model described in the previous section shows the same trend of equity participation found in theoretical and empirical literature,[30] that is, the convenience of investing more heavily in equities in the early years of working life and then gradually decreasing as one approaches retirement age.

108. Assuming some home bias constraints, the scarce availability of time series data in Colombia is a severe limitation for creating optimal long-term portfolios that replicate the optimal investment trajectory for a representative individual. While portfolios derived from optimization models for the case of Colombia are subject to errors, at least provide an unbiased estimate of the long term portfolio composition that optimizes the participants pensions (Figure 3).

109. With the support from financial and economic theory it is possible to design portfolios that follow long-term logic that are Pareto superior to those that portfolio equilibrium is derived from a purely competitive process. As suggested by Basak and Makarov (2009) and Castañeda and Rudolph (2010), a manager's interest in reaching prominent positions in the ranking of returns, and the use of the average return of the industry for the calculation of the reference portfolio, leads to short-term asset allocations that are sub-optimal for contributors interested in receiving good pensions at retirement age.

[30] See, for example, Campbell and Viceira (2002) and Hinz et al. (2010).

110. This paper proposes a gradual strategy of convergence to a competition based on an exogenous long term benchmark portfolio. As a way of recognizing the weaknesses of the model and the bias of the average of the industry, this paper proposes the creation of a reference portfolio to be calculated as the weighted average between an exogenously defined long-term benchmark and the average industry return, allowing the relative share of the long-term benchmark to keep increasing (and therefore decreasing the relative share of the average industry return) to the extent that the availability and quality of data improves.

111. While the data is still insufficient (which might be the case in the next decade or so), the long-term exogenous benchmark should be derived from a combination long term empirical findings and consistency with the preliminary outcome of the optimization model. It is important to emphasize that the benchmark be replicable by pension funds, and therefore should be based on market indices that are transparent and replicable for participants. It is proposed that a benchmark portfolio is created based on four indicators:

a. Foreign Equities: MSCI *world* (unhedged)
b. Local Equities: *Colcap*
c. Long-term Inflation-Indexed Government Bonds: To be built
d. Index of fixed income liquidity: IDTEX

112. The *MSCI World* is a global equity index that is well diversified and can be replicated.[31] The *Colcap* is a local equity index that is replacing the *IGBC*, due to methodological problems of the latter.[32] The *Colcap* is the only stock index replicable in Colombia. The *IDTEX*, is a liquidity ratio of prices of fixed income instruments issued by the government, and may serve as a benchmark to evaluate other fixed income instruments, including corporate and mortgage bonds. It is expected that other medium-term fixed income indices may emerge in the future. The only index necessary to construct is a long-term fixed income index that is indexed to inflation.

113. Since Colombian interest rates are typically higher than the ones in developed economies, in "normal conditions" it is not attractive for local investors to invest in foreign sovereign bonds; hence the proposed portfolio excludes foreign fixed income (bonds). The primary reason for the lack of interest in these instruments is the strong rate differential in Colombia with respect to developed economies. The risk is captured by an open exposure to hard currencies. Reinhart and Rogoff (2010) suggest that banking crises are typically followed by sovereign default. The recent rescue packages for Greece, Ireland, and Portugal are just one more indication of the potential and relatively unknown risks that exist in the sovereign debt market, especially after the recent financial crisis. Foreign fixed income can be introduced in the Colombian pension portfolios in a later stage, when risks become more identifiable.

114. For implementation purposes, the pension portfolios identified in Figure 3 can be built as a linear combination of two of the three portfolios that are offered in the Colombian pension system. It is important to highlight that individuals can only invest in the aggressive portfolio through active selection (not by default). Based on the

[31] More specificity of these indices will be required in order to error track these investments over time.
[32] Colcap is a market capitalization index and the IGBC was a market volatility index. The IGBC was heavily affected by changes in few securities. At some point for example, the price Ecopetrol represented almost one third of the composition of the index.

results presented by the model in the previous section, and taking into consideration the long-term empirical regularities, it is proposed to create a preliminary long-term benchmark with the following characteristics:[33]

Example of Benchmark for Colombian Pension Funds (Percentage of the Fund)

	Aggressive	Moderate	Conservative
Colcap	30	20	5
MSCI World (hedged)	50	40	15
IDTEX	10	10	30
TESLPUVR[1]	10	30	50

[1] Index of long-term government bonds indexed to inflation (to be buit)

115. These percentages should be understood as referential, but in the opinion of the authors, they correspond to a reasonable approximation of the Colombian pension system. Except for methodological revisions every few years, the weight of these ratios should remain fixed over time.

116. The proposed methodology is an important difference from the current *synthetic portfolio* used to benchmark the Colombian pension funds. Currently, the weights of the *synthetic portfolio* reflect the effective participation of the pension funds in that class of assets. Since these weights change over time there is practically no influence of this benchmark over the investment decisions of the pension fund management companies. Probably the only conflicting factor of the *synthetic portfolio* is the portfolio valuation of Treasury notes which is not easily replicable in the market (and whose composition is only known ex-post). The synthetic portfolio adds an unnecessary factor of uncertainty to the pension fund managers, and it does not add much value to fulfill the long term expectations of the contributors.

117. In the future, the benchmark weights should be determined by experts. In order to ensure independence in the design of a long-term portfolio benchmark, it should be done so by a high level commission that operates on a permanent basis. The members of the commission should represent the long term interests of the contributors in the system, and clear terms of reference should specify their mandate. In order to ensure independence, the members of this high level commission could be appointed by the government, and eventually ratified by the Parliament, but most importantly the members of the high level commission should not have a conflict of interest with the government or the private pensions industry. Based on technical studies and long-term considerations, this group should design long-term benchmark *p*ortfolios.

118. From a governance standpoint, the long-term benchmark should not be set by the government, the Pension Supervisor, or the pension industry. This type of work requires a kind of capacity that probably is not present in any of these institutions. While pensions industry has developed some level of market knowledge, their focus tends to be more tactical than strategic. Even more importantly, all of these institutions have insurmountable conflicts of interest when designing these portfolios.[34] The

[33] Denmark's ATP built their lifecycle model (SP) based on three funds of funds.

[34] The experience in the USA previous to the crisis is that target date funds competed by returns among them by increasing the equity exposure. In the case of Lithuania, instead, each pension fund designed

government, for example, has the incentive to use pension funds for debt financing, or to support various government social programs with low rates of return. If the benchmark were designed by the government, it is very likely that this would result in very low pensions in the future. The pension's supervisor (*Superintendencia Financiera*) should not have other duties apart from supervising the industry - otherwise it would not have sufficient distance from investment decisions taken by the pension funds. The industry, meanwhile, has no incentive to align its strategies with the long-term, especially as it generates conflict with short-term returns.

119. Under criteria of transparency and disclosure, it is possible to replicate the mandates of independency of the high level commission. Since legal restrictions can be an impediment for the commission of experts to determine the long term benchmark portfolio, it is possible simulate similar results with an advisory board. Think of a framework in which the President of the Republic creates a high level commission to design this lifecycle benchmark portfolio. Based on a number of studies, the high level commission submits a proposal to the government on the long term benchmark portfolio and a justification for such portfolio allocation. The proposal is submitted with high level of transparency and disclosure to the Minister of Finance. Although the government may or may not adopt the proposal, if decides not to adopted it, it should respond formally (and also with high level of transparency) to the members of the commission justifying the reasons for not accepting their recommendations.

7.2 Building the transition

120. In order to ensure a smooth transition and undesirable changes in the domestic asset prices, the movement from a single portfolio to a multiple portfolios should not generate abrupt changes in portfolio allocation. The changes should be gradual and therefore the reference portfolio parameters should be slowly adjusted to the long-term equilibrium. The domestic asset prices should not be greatly affected to the extent that these changes are expected.[35]

121. It is important to design a transition towards long-term parameters to ensure that the price of financial instruments suffers no abrupt changes. At the time of the creation of multiple portfolio options, the regulations should allow financial instruments to be transferred from one fund to another (within the pension fund management companies) without selling titles in the market.

122. The Ministry of Finance should propose a structure of public debt issuances consistent with the needs of pension funds according to the benchmarks and consistent with the transition process towards a long-term equilibrium. The local fixed income exposure in the transition should be defined in terms of *float* and later assume organic growth from historic simulations. In this regard, the availability of long-term inflation index bonds should be a priority, as the fixed income portfolio benchmark is likely to have a large component of these instruments.

different portfolios with different risk exposures, and consequently individuals were unable to make meaningful comparisons.

[35] Although it was not possible to obtain information on the influence of pension funds on asset prices, the growth in equity participation in the local market in recent months should be analyzed more thoroughly. It should not be ruled out that pension funds have affected the price of equity assets, which would be worrying.

VIII. Minimum return guarantee

123. It is well known that minimum return guarantee entails a convergence of portfolios (herding effect). The problem is not the similarity of portfolios, but that they tend to converge towards sub-optimal portfolios.

124. Minimum return guarantee is a "second best" mechanism in order to protect contributors from unscrupulous managers or from uninformed contributors. There are other alternative proposals for minimum return guarantee frameworks, for example the traffic light framework [see, for example, Rudolph and others (2010)], but these are still in incipient development stages, which, in the absence of a better alternative it is appropriate to continue with existing frameworks.

125. The alternative of completely eliminating minimum return guarantee, without enhancing the supervisory approach, has been tested elsewhere and has resulted in risk allocations that are not easily identifiable by individuals, and divergence in the managed risks. In the case of Lithuania, for example, the pension funds began to offer funds that were unique to each other, and in many cases, with much higher risk levels than reasonable for an obligatory pension system. At the time of the crisis some of these funds suffered losses exceeding 60 percent of the value of the fund, since the risks had not been adequately measured. Despite some recoveries in the aftermath of the crisis, it was a sense of lack of clarity about the objectives and the need of such risky portfolios.

Table 1: Design for the minimum return guarantee

Country	Reference Portfolio and Boundaries	Evaluation Period	Evaluation Frequency	Portfolio Disclosure
Argentina (2008)	Min (70% of the PPRS, PPRS - 2%)	12 months	Monthly	--
Colombia	A 70% of the PPRS B 70% of the RPS 70% of the BVC index return 70% of the S&P 500 return Minimum Return = (A+B)/2	36 months	Monthly	30 days
Chile	High Risk Funds = Min (50% of PPRS, PPRS - 4%) Conservative Funds = Min (50% of the PPRS, PPRS - 2%)	36 months	Monthly	10 days
El Salvador	Min (80%PPRS, PPRS-3%)	12 months	Monthly	
Uruguay	Min (PPRS-2% , 2% real)	12 months	Monthly	
Bulgaria	Min (60 % of the PPRS, PPRS -3 %)	24 months	Quarterly	quarterly
Croatia	If PPRS>0, Min (1/3 PPRS, TD), If PPRS<0, Min (3 PPRS)	12 months	Quarterly	
Poland	Min (50% of the PPRS, PPRS - 4%)*	36 months	Bi-Annually	bi-annually
Romania	Min (PPRS-4%, 50% PPRS)	24 months		
Slovakia	Conservative Fund = Min (90% ARS1, ARS1 - 1%) Balanced Fund = Min (70% ARS1, ARS1 - 3%) Growing Fund = Min (50% of ARS1, ARS1 - 5%)	24 months	Daily	bi-annually

Sources: SBC, SP, KNF, NBS, PPSSC, CFSSA, CSSPP, HANFA.
* The weights are capped at 15% and shares of the remaining funds are increased proportionally until they reach 100%.
PPRS: Weighted Average of the returns of the system; ARS1 is the arithmetic mean of the average annual moving of changes in the daily values of competing pension funds, RPS: Synthetic Portfolio Performance; BVC Index: Colombian Stock Market Index; TD: Central Bank discount rate

126. As shown in Table 1, the minimum return guarantee system of the individual capitalization system in Colombia is in many ways more restrictive than other countries with similar systems. In particular, monthly evaluations are required by

the *Superintendencia Financiera* while in other countries these practices occur quarterly or semi-annually. While from a long-term perspective having monthly reports does not make much sense, the change in frequency of evaluations from quarterly to monthly occurrences happened a few years ago as a response to erratic pension fund investment behaviors approaching the time of measurement. Some pension fund made abuse of last minute investments in order to show better quarterly results. This behavior was inconsistent with serious asset management focused on the long-term objectives of the pension system. The introduction of risk-based supervision frameworks, accompanied with greater responsibility on behalf of the pension fund management boards, would prevent these types of situations from happening again.

127. This paper proposes to measure the minimum return guarantee against the reference portfolio (PR), which is the weighted average of the long-term benchmark and the industry average return and should grant higher degrees of freedom to higher risk funds. In particular it aims to increase the breadth of the band of aggressive and balanced portfolios, from the current 30 percent to 50 percent. Since it is possible to think of future returns in a range between 0 and 5 percent, it is recommended to impose the minimum of the range to be between a percentage of the reference portfolio return and a reference portfolio return minus a determined percentage. The proposal suggest establishing a maximum difference of 4 percent for the benchmark for aggressive and balanced funds, and 2 percent for conservative funds:

Min (50% PR, PR-4%), agressive and balanced portfolios
Min (70% PR, PR-2%) conservative portfolio

128. The possibility of moving towards quarterly, semi-annual or annual evaluations depends on the progress that the pension funds can make in terms of the framework definition of responsible risk management and the progress that the Financial Superintendent makes in risk-based supervision. As an incentive for pension fund managers to improve their risk management frameworks, they will be subjected to an annual or semi-annual evaluation, while those who do not improve will be subject to monthly evaluations. The implementation of a risk based supervision approach is the subject of a different study.

APPENDICES

Appendix A. The portfolio selection model

1. In this section we discuss the portfolio selection model used to make recommendations in the paper. It is worth mentioning that the problem studied in this section is mainly normative, in the sense that the analysis aims to characterize how mandatory pension funds (FPO) should be invested in Colombia, as opposed to a more positive analysis, that is, aimed at explaining the reasons why certain investment decisions are adopted by the pension fund management company in relation to those adopted by the FPO.

- preliminary considerations
- The model developed is the result of considerations that include three dimensions:
- the objectives of the Colombian pension system,
- the restrictions surrounding its operation, and
- the investment opportunities available for the FPOs[36]

129. The need to identify the objectives of the Colombian pension system is a fundamental requirement to determine, at a later stage, an asset allocation that adequately serves these objectives. In this sense, Barr and Diamond (2006) define the main objective of contributory pension systems as delivering sufficient income to individuals during the payout phase, leaving other objectives such as economic growth in the background. In the case of Colombia, the aforementioned objective is compatible with the provisions of regulation (see, for example, Law 100, from 1993) and so it will be adopted in the future.

130. Moreover, the coexistence of a distributive pension system with a defined contribution, as in Colombia, causes additional considerations to be weighed. In particular, the investment of pension funds in public debt titles constitutes an important component of the domestic demand for government titles, which makes it necessary to evaluate the feasibility of proposing an asset allocation far from the actual situation. A similar situation happens with domestic equity investments. In this case, there are concerns related to the increasing size of the FPOs and the local stock market for the possible pressures that the said demand may cause for domestic asset prices.[37]

131. Beginning with the aforementioned considerations, along with others related to the tolerance of pension fund losses and fiscal interest, it is possible to identify two constraints surrounding the operation of the Colombian pension system:

- The optimal portfolio should be such that the annuitization risk is limited. At the same time, it should safeguard the fiscal interest associated to the event that a significant proportion of the population fails to finance a minimum pension equal to the existing minimum wage at the time of retirement.

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[36] The description of FPO investment opportunities is discussed in detail in Appendix B.
[37] Strictly speaking, the relevant variable is the percentage of outstanding shares that are effectively available to be negotiated, without the purpose of gaining control of society.

- The need for public financing and the existence of the distributive component in the Colombian pension system suggest that the optimal portfolio ought not to involve an investment higher than 50 percent of the debt portfolio of the central government (that is, excluding the debt issued by the Central Bank of Colombia).

<u>A portfolio selection model</u>

132. In response to the objectives and restrictions previously mentioned, we next develop a life cycle model for portfolio selection that a representative individual faces. The model in question is in line with the tradition indicated by Merton (1969, 1971) and Samuelson (1969) four decades ago, one which has undergone significant improvements and refinements during the last 20 years.[38] A good revision of this literature can be found in the monograph by Campbell and Viceira (2002).

133. The proposed model is based on an optimization problem that summarizes the objectives of the pension system through an objective function of expected utility and attempts to capture, through an algebraic representation, the essence of the objectives, constraints and opportunities for investment that characterize investment decisions in the Colombian pension system. In particular, the proposed function corresponds to:

$$U(X_T) := \mathbb{E}[e^{-\rho T} u(X_T/P_T - \eta \bar{c}_T)], \tag{A.1}$$

where $\mathbb{E}[\cdot]$ corresponds to the mathematical outcome, $\rho \geq 0$ is a subjective discount factor, $u(\cdot): \mathbb{R}_+ \to \mathbb{R}$ is an instantaneous utility function (strictly) increasing and concave, that has as its argument the amount of the pension that is possible to finance (X_T/P_T), which in turn depends on the accumulated amount in the individual account $(X_T \geq 0)$ and the price of unitary annuity at the time of retirement $(P_T \geq 0)$. Additionally, the objective function considers the fraction $\eta \in [0,1]$ of the minimum wage $\bar{c}_T \geq 0$ as the minimum level of the pension.

134. The adoption of the function in (A.1) is founded on capturing the objective of the pension system as much as the fiscal interest. In the first case, this happens by making explicit the relationship that exists between the accumulation period and de-accumulation of funds, which is reflected in the argument $u(\cdot)$ that corresponds to the value of the pension that would be obtained at retirement (X_T/P_T), instead of a mere amount accumulated in the individual account (X_T). This is achieved by capturing the risk an individual faces at the time of transforming the accumulated savings into a stable annuity during the period of destocking.

135. One of the direct implications of the proposed specification is that the optimal portfolio obtained, defined as one that maximizes $U(X_T)$, must take special care to

[38] See, for example, Cox and Huang (1989, 1991), Karatzas et al. (1987), Dybvig and Huang (1988), He and Pearson (1991), Karatzas et al. (1991), Bodie et al. (1992), Cvitanic and Karatzas (1992), Kim and Omberg (1996), Bertaut and Haliasos (1997), Brennan et al. (1997), Cuoco (1997), Heaton and Lucas (1997), Campbell and Viceira (1999, 2001), Barberis (2000), Xia (2001), Brennan y Xia (2002), Wachter (2002), Detemple et al. (2003), Brandt et al. (2005), Cocco et al. (2005), Liu (2006), Detemple and Rindisbacher (2010), Buraschi et al. (2010), among others.

cover increases in the cost of financing a pension unit (P_T), since in this context it corresponds to a risk-free asset.[39]

136. Fiscal interest, for its part, is taken into consideration when measuring the instantaneous utility that the individual gets for the pension that is capable of financing (X_T/P_T), over the fraction ($\eta \in [0,1]$) of the level of minimum wage at the time of withdrawal ($\bar{c}_T$). The individual's motivation, in this case, comes from obtaining at least $\eta\bar{c}_T$ at retirement, where the amount is considered the minimum amount acceptable, since $u(x) = -K$, for $x < 0$, with $K > 0$. Because of this, as in the previous case, there will be a motivation for the optimal portfolio to procure coverage for the increase of minimum wage over time, because otherwise, the individual runs the risk of obtaining a pension below the said amount, and in consequence, a punishment of K.

137. The latter, in turn reduces the fiscal impact in the case that these individuals are entitled to the minimum pension, or rather, have a high probability of moving into the distribution system, since the optimal portfolio will avoid the associated costs of said event if at all possible.

138. For purposes of considering constraints like those previously mentioned, the portfolio problem determines the search for the optimal [that is, one that maximizes $U(X_T)$], and those that satisfy the condition:

$$\pi_t \in \Pi_t \subseteq [0,1]^d, \tag{A.2}$$

where $d \geq 1$ is the number of risky assets (in the sense that their future return is unknown) that can invest their resources in pension funds, π_t corresponds to the weighted vectors that make up the investment portfolio in the instant $t \in [0, T]$, and Π_t represents the set of admissible investment portfolios, and includes possible restrictions for investment in FPOs in certain asset classes, or specific instruments, valid in t (for example, $\pi_{it} \in [\underline{\pi}_{it}, \bar{\pi}_{it}], \forall t \in [0, T]$, for some $i \in \{1, 2, \dots, d\}$). In consequence, the percentage invested in deposits over time, which, unlike risky assets d, have a known return t, given by $1 - \pi_t^{'}\mathbf{1}_d$,, where (') denotes the transposition of vector operation and $\mathbf{1}_d := (1, \dots, 1) \in \mathbb{R}^d$.[40]

b. A parametric model

139. Next, we will present a parametric model that fits within the general guidelines of the portfolio selection model previously described previously. The parametric version considers the following elements:

i. *Continuous Time*. Time evolves continuously throughout the investment horizon, $t \in [0, T]$, where $T > 0$ corresponds to the time that the representative individual receives their pension.

[39] According to Wachter (2003), the the long-term bond that matures at the time of retirement is the risk-free asset for an individual that is infinitely risk averse. The general result (for any level of risk aversion) has been recently shown by Detemple and Rindisbacher (2010).

[40] It is worth highlighting that the characterization described accommodates restrictions of short-term volatility defined according to a reference asset. In particular, Cuoco et al. (2008) and Pirvu (2007) have shown that a VaR (Value-at-Risk) restriction is equivalent to a quantitative portfolio restriction, and can consequently be captured by the set of admissible portfolios (Π_t).

ii. *Financial Market.* The financial market is comprised of $d \geq 1$ risky assets (that is, their price in $s > t$ is unknown in the moment t) and by a bank account. The price of a risky asset $i \in \{1, \dots, d\}$ is given by: $dS_{it}/S_{it} = (\hat{\mu}_{it} - \delta_{it})dt + \sigma_{it}'dW_t$, where S_{it} is the price in the instant t, $\hat{\mu}_{it}$ is the instantaneous expected return to be received by the individual, δ_{it} is the instantaneous dividend rate, $\sigma_{it} \in \mathbb{R}^d$ is the asset volatility, and $dW_{it} \in \mathbb{R}^d$ is a Brownian motion (normal) shock with mean zero and variance dt in each coordinate. In turn, the price of the bank account evolves according to $dB_t/B_t = \hat{r}_t dt$, where $B_0 = 1$ y $\hat{r}_t$ is the interest rate received between t and $t + dt$ for the individual.[41]

iii. *Labor income process.* The growth rate of the contribution made by the individual to the individual account given by $de_t/e_t = \mu_t^e dt + \sigma_t^e{}'dW_t$ where μ_t^e is the expected growth rate and σ_t^e is the input sensitivity to the shock affecting risky assets.

iv. *Unitary annuity cost.* The price in t of a unitary annuity with deferred payments until $t = T$ corresponds to $P_t^T = \int_T^\infty e^{-f_{t,s}} B_t^s ds$, where $f_{t,s}$ is the mortality force of an individual at the age of $s + 20$ that is alive at t, and B_t^s is the price in t of a bond (risk-free) that pays \$1 in $s \geq T$.

v. *Minimum Income.* The minimum income at the moment of divulgement is given by a constant $\bar{c}_T$ (see Appendix B).

140. Based on these parametric assumptions, and denoting as X the value of the individual account of the representative individual, the problem of *dynamic* portfolio selection presented by the case of two assets (that is, $d = 1$) can be written as: [42]

$$\max_{\pi \in \mathcal{A}} \mathbb{E}_0[e^{-\rho T} u(X_T/P_T - \eta \bar{c}_T)] \tag{A.3}$$

$$\text{where } \mathcal{A} := \begin{cases} dX_t^\pi = X_t^\pi[(1 - \pi_t)\hat{r}_t dt + \pi_t(dS_t/S_t + \delta_t)] + e_t dt; \\ X_0 = x > 0;\ X_t^\pi \geq 0, \forall t \in [0, T];\ X_T^\pi \equiv X_T;\ \pi_t \in \Pi_t;\ U(X_T) < \infty \end{cases}$$

141. The problem in question presents a major challenge for the dynamic portfolio selection dilemma of standard portfolios [Merton (1971), Cox and Huang (1989), Karatzas et al. (1987)]: quantitative portfolio restrictions, which are reflected in the set of admissible portfolios (Π_t).

142. It should be noted that under the assumptions of the parametric model it is possible to characterize the set of admissible portfolios as:

$$\Pi_\mathrm{t} := \{\pi_t \in [\underline{k}_t := max(\underline{\pi}_t, 0), \overline{k}_t := min(\overline{\pi}_t, 1)]\}.$$

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[41] See below the difference between the collected variables $(\hat{\mu}_t, \hat{r}_t)$ and those effectively prevailing in the market (μ_t, r_t).

[42] The results are shown for the case of two assets only not to complicate this notation further. The extension of the case with two or more 'risky assets' is straightforward; see, for example, Detemple et al. (2005).

where $\underline{\pi}_t$ and $\overline{\pi}_t$ are the quantitative investment limits. For purposes of incorporating these quantitative conditions we will apply the method proposed by Cvitanic and Karatzas (1992), consistent with the inclusion of dynamics of existing assets in the financial market, the Lagrange multiplier associated to the restrictions contained in Π_t.

143. In particular, this involves setting $(\hat{\mu}_t, \hat{r}_t)$ as follows: $\hat{\mu}_t \coloneqq \mu_t + \lambda_t + \phi(t, \lambda_t)$ and $\hat{r}_t \coloneqq r_t + \phi(t, \lambda_t)$, where (μ_t, r_t) corresponds to the expected return and the interest rate (instantaneous) prevailing in the market, λ_t is the multiplier associated with the restriction that strengthens the optimal investment strategy contained in Π_t, and $\phi(t, \lambda) \coloneqq sup_\pi\{-\pi\lambda\} = \underline{k}\lambda^- + \overline{k}\lambda^+$ is the support function of the multiplier λ, with $\lambda^\pm \coloneqq max\,(0, \pm\lambda)$, which also satisfies the property: $\phi(t, \lambda) + \pi\lambda = 0$.

144. The method in question rests in the analysis and solution of a portfolio problem without quantitative portfolio restrictions (that is, $\Pi_t = \mathbb{R}$), where the coefficients of the market are the receivables $(\hat{\mu}_t, \hat{r}_t)$, instead of the effectives (μ_t, r_t). The key point is that the solution of this problem coincides exactly with that of the portfolio problem with quantitative restrictions, when λ is effectively the Lagrange multiplier that forces the investment strategy to be contained in Π_t. The intuition of the result is given by the relationship between equity risk rewards in both cases, $\mu_t - r_t + \lambda_t = \hat{\mu}_t - \hat{r}_t$, and the direct impact that the latter quantity has in the scale of the destined investment of risky assets. [43]

145. To facilitate the exposition, we identify next the solution of the portfolio selection problem in a simplified environment, one that corresponds to the version employed in the report.

Result A.1: Consider the parametric model developed above and assume that:

- The function $u(\cdot)$ is such that $-x \cdot u''(x)/u'(x) = R > 0, \forall x > 0$,
- the coefficients $(\mu_t - r_t, \sigma_t^e, \sigma^{\bar{c}})$are constant,
- the instantaneous interest rate (r_t) satisfies the dynamics $dr_t = \kappa_r(\bar{r} - r_t)dt + \sigma_r dW_t^Q$, with Q as the risk-neutral measure, and the coefficients $((\mu_t^e, \sigma_t, f_{t,s}))$ are deterministic functions of time.

Thus the optimal portfolio is given by:

$$\pi_t^* = \frac{(\mu + \lambda_t)}{R\sigma_t^2}\left(1 + f_t^{HK} - f_t^{\bar{c}}\right) + \frac{\sigma^{\bar{c}}}{\sigma_t^2} f_t^{\bar{c}} - \frac{\sigma^e}{\sigma} f_t^{HK} + H_t^{r,\lambda}$$

where f_t^x corresponds to the fraction of the pension fund for financing: 1) a pension of at least minimum wage $(x = \bar{c})$, 2) an efficient portfolio in the sense of mean-variance $(x = MV)$, and 3) a portfolio that 'reverses' the effect of human capital in the aggregated investments of the individual, respectively. For its part, $H_t^{r,\lambda}$ corresponds to the 'coverage' component motivated by fluctuations in interest rate *(r)* and the multiplier *(λ)* entailed in quantitative restrictions contained in the set Π_t and its expression is provided in the proof.

[43] In particular, the value of λ is adjusted by reducing (increasing) the risk reward when the optimal investment exceeds (is inferior) the limit $\overline{\pi}_t$ $(\underline{\pi}_t)$.

146. Result A.1 characterizes the optimal investment portfolio in an environment that, despite its simplicity, is capable of capturing all the relevant elements for the pension fund case.

147. In particular, the environment in question considers the uncertainty of the expected changes in price of annuities (P_T) and the minimum wage valid at the time of pension ($\bar{c}_T$), both produced by the uncertain fluctuations in the instantaneous interest rate.[44] When the evolution of the interest rate is unknown, the optimal portfolio takes into account the increases and reductions in the expected cost of financing the three motives of investment identified in Result A.1: minimum wage, human capital, and mean-variance efficiency.

148. The optimal portfolio in the expression is broken down in two parts: a component of mean-variance efficiency and a component of coverage motivated for three reasons: to cover the increased cost of the portfolio that finances minimum income, to cover the loss of value that human capital may suffer, and to cover adverse changes in the opportunity set.

149. One aspect worth highlighting is that investing in assets that are mean-variance efficient is subject to the financial sufficiency to cover (or finance) the minimum wage at the time of retirement.

150. *Proof of Result A.1.*

We present the proof of the A.1 outcome for the general case of $d > 1$ risky assets, where the *admissible* portfolios belong to the set:

$$\Pi_t = \begin{Bmatrix} \pi_t \in \mathbb{R}^d : \pi_{it} \in [\alpha_{it}, \beta_{it}], \alpha_{it} \in [0, \beta_{it}], \beta_{it} \in [\alpha_{it}, 1], i \in \{1, \dots, d\}, \\ \sum_{i=1}^{d} \pi_{it} \leq 1 \end{Bmatrix}$$

where $(\alpha_{it}, \beta_{it})$ are (progressively measurable processes) whose values are given by regulation.[45] It should be noted that $\pi_t \in \Pi_t$ implies that $X_t^\pi \geq 0$, since the contributions are non-negatives *(*$e_t \geq 0$*)*, and the restrictions contained in Π_t limit the possibility of the individual account going into debt; as much as the short sale, as going into debt with a risk-free asset. Additionally, based on the definition Π_t the support function is given as$\phi(\lambda_1 + \lambda_2) \coloneqq sup_{\pi \in \Pi}\{-\pi^{'}(\lambda_1 + \lambda_2)\} = \min(\beta' \lambda_1^{-} - \alpha' \lambda_1^{+}, -\min(\lambda_2))$; Cvitanic y Karatzas (1992, §14.7-§14.9). *Proof of Outcome A.1:* Consider the portfolio problem in a fictitious market where the coefficients of the assets are: *(*$\hat{\mu}_t, \hat{r}_t, \sigma$*)*, with $\hat{\mu}_t \equiv \mu + \lambda_t + \phi(t, \lambda_{1t} + \lambda_{2t})$, with $\hat{r}_t \equiv r_t + \phi(t, \lambda_{1t} + \lambda_{2t})$ as the solution of the indicated dynamics in the outcome $\pi_t \in \mathbb{R}$. Given that $\phi(t, \lambda_1 + \lambda_2) + \pi^{'}(\lambda_1 + \lambda_2) = 0$, for any $(t, \omega) \in [0, T] \times \Omega$, then the solution to this problem coincides with the solution to the problem where $\pi_t \in \Pi$; Cvitanic and Karatzas (1992), Karatzas and Shreve (1998, §6).

[44] If this element were absent, both elements would be irrelevant for the optimal portfolio policy, since the future trajectory would be completely predictable, and the hedging motive would translate directly in an increased investment in bank deposits (to finance the future minimum wage), while the annuity price would be irrelevant, because it would be a constant value at the time of retirement.
[45] Note that the set in question accommodates the restrictions of the VaR; Cuoco et al. (2008), Pirvu (2007), defined according to a reference asset.

Later, the problem (A.5) can be written as [Cox and Huang (1989, 1991), Karatzas et al. (1987)]:

$$\max_{X_T \in \mathcal{A}^\pi(x,\Pi)} \mathbb{E}_0[e^{-\rho T} u(X_T/P_T - \eta \bar{c}_T)]$$

$$\mathcal{A}^\pi(x,\Pi) := \begin{cases} \mathbb{E}[\hat{\xi}_T X_T] \leq x + \mathbb{E}\left[\int_0^T \hat{\xi}_t e_t dt\right] \\ X_0 = x > 0;\ U(X_T) < \infty \end{cases}$$

from where you have $X_T^* = P_T\{J(y\hat{\xi}_T e^{\rho T}) + \eta \bar{c}_T\}$, $\hat{\xi}_T := \exp\left(-\int_0^T \left(\hat{r}_t + \frac{1}{2}\hat{\theta}_t{}'\hat{\theta}_t\right) dt - \int_0^T \hat{\theta}_t{}' dW_t\right)$ is the (only) stochastic discount factor compatible with the absence of arbitration, $\hat{\theta}_t := \theta_t + \sigma_t^{-1}\lambda_t$ is the price for market risk [with $\theta_t := \sigma_t^{-1}(\mu - r_t)$], and $J(\cdot) := u'^{-1}(\cdot)$.. Additionally, we have the optimal portfolio that finances the optimal amount of the desired final accumulated wealth X_T^*, a quantity from:

$$\hat{\xi}_t X_t^* = \mathbb{E}_t\left[\hat{\xi}_T X_T^* - \int_t^T \hat{\xi}_s e_s ds\right]..$$

As it is well known, the optimal portfolio can be obtained from the volatility of optimal wealth [see dynamics of X_t^* in (A.3)]. Then, in the case of preferences with constant relative risk aversion [that is, $-x \cdot u''(x)/u'(x) = R > 0]$], a direct application of Itô's Lemma yields the following expression for portfolios associated with the optimal wealth process:[46]

$$\pi_t^* = (R\sigma_t\sigma_t')^{-1}(\mu + \lambda_t)\left(1 + f_t^{HK,\lambda} - f_t^{\bar{c},\lambda}\right) + \sigma_t'^{-1}\sigma_{\bar{c}} f_t^{\bar{c},\lambda} - \sigma_t'^{-1}\sigma^{e'} f_t^{HK,\lambda} + H_t^{r,\lambda},,$$

where $f_t^{HK,\lambda} = \mathbb{E}_t\left[\int_t^T \hat{\xi}_s e_s ds\right] \div \hat{\xi}_t X_t^*$, $f_t^{\bar{c},\lambda} = \mathbb{E}_t[P_T \hat{\xi}_T \eta \bar{c}_T] \div \hat{\xi}_t X_t^*$, and

$$H_t^{r,\lambda} = \left(\hat{\xi}_t X_t^*\right)^{-1} \sigma_t^{-1} \mathbb{E}_t\left[\hat{\xi}_T \eta \bar{c}_T \left\{H_t^{P,\lambda} - H_{t,T}^{\hat{\xi}}\right\} + \hat{\xi}_T J\left(y\hat{\xi}_T e^{-\rho T}\right)\left\{H_t^{P,\lambda} - P_T H_{t,T}^{\hat{\xi}}(1 -\right.\right.$$
1/R)+tT..sesHt,s..ds

with $H_t^{P,\lambda} = \mathcal{D}_t(P_T)$, $H_{t,T}^{\hat{\xi}} = \int_t^T \left(\mathcal{D}_t(\hat{r}_s) + \hat{\theta}_s \mathcal{D}_t(\hat{\theta}_s)\right) ds + \int_t^T \mathcal{D}_t(\hat{\theta}_s) dW_s$, and $\mathcal{D}_t(\cdot)$ as the derivative by Malliavin;[47] Detemple et al. (2005), Nualart (2006).

151. Similarly, note that it is possible to identify the multiplier associated with restrictions from the characterizations already presented. In particular, note that $\sum_{i=1}^d \pi_{it} \leq 1$, $\pi_{it} \in [\alpha_{it}, \beta_{it}]$ requires that (see Detemple and Rindisbacher (2005) for more detailed work in a similar problem):

46 See, for example, Detemple et al. (2003, 2005), Detemple and Rindisbacher (2005).

47 The Malliavin derivative is a generalization of the concept of the traditional derivatives that extends the concept to functions that depend on the paths of Brownian motion. In the same way the ordinary derivative measures the local change in the function, compared to a local change in the underlying variabl e, the Malliavin derivative measures the change in the function (which depends on the trajectory of W) implied by a small change in the trajectory of W. The interested reader can refer to Detemple et al. (2005) for a brief introduction to this operator, and Nualart (2006) for an exhaustive treatment.

$$(\theta_t + \sigma_t^{-1}\lambda_t) \geq \left(1 + f_t^{HK} - f_t^{\bar{c}}\right)^{-1} R\left(-\phi(\sigma_t) - \sigma_{\bar{c}} f_t^{\bar{c}} + \sigma^e f_t^{HK} - H_t^{r,\lambda}\right)$$
$$(\theta_t + \sigma_t^{-1}\lambda_t) \leq \left(1 + f_t^{HK} - f_t^{\bar{c}}\right)^{-1} R\left(\phi(\sigma_t) - \sigma_{\bar{c}} f_t^{\bar{c}} + \sigma^e f_t^{HK} - H_t^{r,\lambda}\right)$$
with $\phi(\lambda_1) = \beta'\lambda_1^- - \alpha'\lambda_1^+$, that is, it requires the multiplier to adjust the risk reward in a way that yields an outcome that easily remains within acceptable ranges.

152. Specifically, give that $\phi(t, \lambda_1) + \pi'\lambda_1 = 0$ is always true (by construction), the multiplier is given by:[48]
$$\lambda_t = \pm\left(-\theta_t + \left(1 + f_t^{HK} - f_t^{\bar{c}}\right)^{-1} R\left(\mp\phi(\sigma_t) - \sigma_{\bar{c}} f_t^{\bar{c}} + \sigma^e f_t^{HK} - \sigma_t{}' H_t^{r,\lambda}\right)\right)^{\pm}$$
Since the solution of the multiplier involves solving a system of "backward" stochastic differential equations (also known as BSDE) whose solution is unknown, we opt to approximate the optimal portfolio through the expression:
$$\pi_t^{*,0} = (R\sigma_t\sigma_t{}')^{-1}(\mu + \lambda_t^0)\left(1 + f_t^{HK,0} - f_t^{\bar{c},0}\right) + \sigma_t^{-1}\sigma_{\bar{c}} f_t^{\bar{c},0} - \sigma_t^{-1}\sigma^e f_t^{HK,0} + H_t^{r,0},,$$
that is, through the modified version of the analytical solution it considers only the contemporaneous effect of quantitative constraints of portfolios contained in a set Π_t.

Derivation of $\boldsymbol{\pi_t^{*,0}}$. In the case where $\lambda_s = 0$, $\forall s > t$ (that is, the restrictions can be active only one at a time), you have $\hat{\xi}_t X_t^* = \xi_t N_t X_t^* = \xi_t X_t^*,,$ where $N_t := \exp\left(\int_0^t (\phi(\lambda_s) + (\theta_s{}'\sigma_t^{-1}\lambda_s + \lambda_s'(\sigma_t{}'\sigma_t)^{-1}\lambda_s)ds) - \int_0^t (\sigma_t^{-1}\lambda_s)' dW_s\right)$. Then, we obtain
$$\xi_t X_t^* = \mathbb{E}_t\left[\xi_T X_T^* - \int_t^T \xi_s e_s ds\right],$$
where
$\mathbb{E}_t[\xi_T X_T^*] = \mathbb{E}_t[\xi_T(P_T\{J(y\xi_T e^{\rho T}) + \eta\bar{c}_T\})] = K_1 P_T \mathbb{E}_t\left[\xi_T^{1-1/R}\right] + \eta\bar{c}_T\mathbb{E}_t[\xi_T P_T],,$
$\bar{c}_T$ is already measurable, with $K_1 = y^{-1/R}e^{-\rho T/R}$, $P_T = \int_T^{T_{\max}} e^{-f_{T,s} - r_T N(T,s) - M(T,s)}\, ds$, $T_{\max} < \infty$ as the maximum age of the individual's life; while
$$\mathbb{E}_t\left[\int_t^T \xi_s e_s ds\right] = \mathbb{E}_t\left[\int_t^T \exp\left(\int_0^s \left(\mu_v^e - r_v - \frac{1}{2}(\theta_v{}'\theta_v + \sigma^e{}'\sigma^e)\right)dv + \int_0^s (\sigma^e - \theta_v)' dW_v\right)ds\right] = \mathrm{HK}_t.$$
Then, $f_t^{HK,0} = \mathrm{HK}_t \div \xi_t X_t^*$, $f_t^{\bar{c},0} = \eta\bar{c}_T\mathbb{E}_t[\xi_T P_T] \div \xi_t X_t^*$, while the terms for $H_t^{r,0}$ we have: $H_{t,T}^{P,0} = \mathcal{D}_t(P_T) = -\int_T^{T_{\max}} e^{-f_{T,s} - r_T N(T,s) - M(T,s)}\, \mathcal{D}_t(r_T) N(T,s) ds H_{t,T}^{\xi} = \int_t^T \left(\mathcal{D}_t(r_s) + \theta_s\mathcal{D}_t(\theta_s)\right)ds + \int_t^T \mathcal{D}_t(\theta_s) dW_s$.

[48]The argument is similar to that used by Detemple and Rindisbacher (2005, pg. 559).

Appendix B. Numerical implementation

153. In this section we present the numerical implementation of the portfolio selection model presented in the previous section. The implementation consists of three stages:

- characterization of the dynamics of the variables of interest (asset returns, contributions to an individual account, minimum wage, etc.)
- estimation/calibration of the parameters involved
- calculating the optimal portfolio for the expected trajectory

a. The dynamics of quantities of interest

154. The following specific dynamics were considered from the base of available assets:[49]

$$dB_t/B_t = r_t dt$$
$$dr_t = \kappa_r(\overline{r} - r_t)dt - \sigma_r dW_{1,t}^Q$$
$$dS_{1,t}/S_{1,t} = \left(r_t + \sigma_1 \hat{\theta}_{1,t}\right)dt + \sigma_1 dW_{1,t}$$
$$dS_{it}/S_{it} + \delta_{it}dt = \left(r_t + \sigma_i\left(\rho_{i,1}\hat{\theta}_{1,t} + \cdots + \rho_{i,i-1}\hat{\theta}_{i-1,t} + \sqrt{1-\rho_i^2}\hat{\theta}_{i,t}\right)\right)dt$$
$$+\sigma_i\left(\rho_{i,1}dW_{1,t} + \cdots + \rho_{i,i-1}dW_{i-1,t} + \sqrt{1-\rho_i^2}dW_{i,t}\right)$$
$$de_t/e_t = \mu_t^e dt + \sigma_e\left(\rho_{e,1}dW_{1,t} + \cdots + \rho_{e,d-1}dW_{d-1,t} + \sqrt{1-\rho_e^2}dW_{d,t}\right)$$

where S_{1t} is the price of a bond index with duration $T_1 \leq T$, $\sigma_1 = N(0,T_1)\sigma_r$, $\sigma_k\hat{\theta}_{k,t} = \mu_k - r_t + \lambda_t$, for $k \in \{1,\dots,d\}$, with $\mu_k > 0$ as the *constant* risk reward associated to the risk factor k. In this case, $W_{1,t}$ corresponds to the interest rate risk factor, while $\{W_{2,t},\dots,W_{d,t}\}$ represents independent risk factors linked to the remaining available assets. For their part, $\rho_i^2 = \sum_{k\in\{2,\dots,d\}}^{i-1}\rho_{i,k}^2$, for $i \in \{2,\dots,d\}$. The correlation between assets i and j, with $i < j$, coming from $\rho_{i,B}\rho_{j,B} + \cdots + \rho_{i,i-1}\rho_{j,i-1} + \sqrt{1-\rho_i^2}\rho_{j,i}$.

155. Note that the dynamics considered imply risk reward, volatilities and constant correlations over time, a modeling alternative that is mainly due to the short data series available, which make non-linearity models less reliable in asset dynamics, like mean reversion, or changes of regimen.

b. Estimation and calibration of parameters

156. In the calibration process diverse market indices were considered in order to allow a reduction in the dimensionality of the problem and greater stability in the solution.[50] At the domestic level, the asset indices COLCAP and IGBC were considered, as well as the fixed income index IDXTES drawn up by Reveiz and León (2008). Additionally, state bond indices were constructed from the interest rate curves

[49] The dynamics in question correspond with the Cholesky decomposition of an asset base with correlated shocks, and on that used by Detemple et al. (2003, §VI.A) and Munk and Sørensen (2010).

[50] This obeys the variance-covariance matrix of returns which tends not to be positively defined when a relatively high number of assets and their returns are highly correlated.

of available zero-coupon bonds. Meanwhile externally, stock indexes such as the S&P500 and the MSCI World Emerging Markets, as well as Lehman Brothers indexes for state and corporate bonds in the U.S. and Euro zone were considered. Additionally, we considered the exchange rate of the peso-dollar (American), peso-yen, and peso-Euro, with the purpose of considering the possibility of the FPO taking positions in foreign assets, either covered or uncovered, in relation to risk fluctuations in the exchanges in investment denominations.

157. In the characterization of returns of the previously indicated indices, the following dynamic for risky assets not *directly* linked to the interest rate was considered:

$$\frac{S_{i,t+\Delta t}}{S_{i,t}} - 1 = e^{\left(\mu_i - \frac{1}{2}\sigma_i{}^2\right)\Delta t + \sqrt{\Delta t}\sigma_i \varepsilon_i} - 1$$

where μ_i is estimated as the value of the 40$^{\text{th}}$ percentile of the distribution of available data, σ_i is the standard deviation of the asset $i \in \{1, \ldots, k\}$, with $k < d$, while ε_i is a random shock that is distributed at normal standards.

158. For its part, in the case of the zero-coupon bonds, the dynamic proposal for the instantaneous interest rate imposes restrictions (non-arbitrage) of the price of these instruments, so that the return between t and $t + \Delta t$ is given by:

$$\frac{B^s_{j,t+\Delta t}}{B^s_{j,t}} - 1 = e^{-r_{t+\Delta t}N(t+\Delta t,s) - M(t+\Delta t,s) + (r_t N(t,s) + M(t,s))} - 1$$

where $N(t,s) \coloneqq (\kappa_r)^{-1}(1 - e^{-\kappa_r(s-t)})$, and
$M(t,s) \coloneqq (\overline{r} - (\sigma_r^2/2\kappa_r^2))(N(t,s) - s + t) - (\sigma_r^2/4\kappa_r)N(t,s)^2)$, with $j \in \{1, \ldots, d - k\}$.

159. Something similar happens with the price of deferred annuity, whose price is given by:

$$P_t^T = \int_T^{T_{\max}} e^{-f_{t,s} - r_t N(t,s) + M(t,s)}\, ds$$

where $f_t \coloneqq f + (1/b)\exp([t-m]/b)$ is the force of mortality of an individual at age $t + 20$,[51] and $f_{t,s} \coloneqq f(s-t) + b(f_t - f)(1 - \exp([s-t]/b)$ is the strength of marginal mortality of an individual of age $t + 20$, that lives until $s + 20$. For interpretative purposes, it should be noted that the expression in question corresponds, simply, to the present value of the bond portfolio that is sufficient to finance a flow of \$1 for the fraction of individuals who remain alive at the time s, according to the life expectancy of the insured group at the instant t and the cost of living.

160. The value of these model parameters was estimated or calibrated based on the following criteria:

f. *Assets not directly linked to the interest rate.* The parameter μ_i is estimated through the average of monthly returns ($\Delta t = 1/12$), while σ_i is estimated from the sample variance of monthly logarithmic-returns [that is, log $(S_{i,t+\Delta t}/S_{i,t})$].

g. *Assets directly-linked to the interest rate.* The parameters $(\overline{r}, \kappa_r, \sigma_r)$, linked directly to the dynamics of the interest rate were calibrated by the curves of available

[51] The functional form of the force of mortality is known in literature as the Gompertz-Makeham type; Carriere (1994). The justification of the parameters comes from the following arguments: the parameter $f > 0$ captures the accidental factors of mortality, while (m, b) are distribution parameters required to be estimated by the available mortality tables.

rates using minimization criteria of the sum of the squares of the residual adjustment results.

h. *Deferred Life annuities.* The parameters (f, m, b) were calibrated by the Chilean mortality tables [Pension Superintendent of Chile (2004)] using the same criteria as above.

i. *Individual Account Contributions*. The dynamics for the contributions were characterized by the following equation: $e_t = e_0 \exp\left(\int_0^t \mu_s^e\, ds - \frac{1}{2}\sigma_e{}^2 t + \sqrt{t}\sigma_e\varepsilon_e\right)$, where σ_e is the volatility of the process, ε_e is a random shock with standard normal distribution $\int_0^t \mu_s^e\, ds \coloneqq \ln(a_0 + a_1 t + a_2 t^2 + a_3 t^3)$. The parameters (a_0, a_1, a_2, a_3) were calibrated by adjusting the trajectory of the expected contributions used by Viceira (2010, pg. 223) for the Chilean case utilizing the previously mentioned criteria, while σ_e utilized the figures used by Munk and Sørensen (2010) for the U.S. economy.

Minimum Wage. The minimum wage was characterized by: $\overline{c}_T = \overline{c}$..

Other parameters. R = 2,5 y T = 35.

c. Optimal portfolio calculation

161. The calculation of the optimal portfolio is preceded by the implementation of the proposed analytic solution in Appendix A, through the simulation of the conditional hopes involved. Based on the developed expression (that is, $\pi_t^{*,0}$), which makes the theoretical model solution numerically implementable, the calculation of the optimal portfolio requires computing the quantities:

$\mathbb{E}_t\left[\xi_T^{1-1/R}\right]$, $\mathbb{E}_t[\xi_T P_T]$, $\mathbb{E}_t\left[\int_t^T \xi_s e_s ds\right]$, $\mathbb{E}_t\left[\xi_T H_{t,T}^{P,0}\right]$, $\mathbb{E}_t\left[\xi_T H_{t,T}^{\xi}\right]$, $\mathbb{E}_t\left[\xi_T^{1-1/R} H_{t,T}^{P,0}\right]$, $\mathbb{E}_t\left[P_T \xi_T^{1-1/R} H_{t,T}^{\xi}\right]$, $\mathbb{E}_t\left[\int_t^T \xi_s e_s H_{t,s}^{\xi} ds\right]$,

problems that are possible to solve by means of simulations, according to the work of Detemple et al. (2003).

162. The quantities in question can be written as *(*with $\vartheta \equiv 1 - 1/R$, and $x_{t,s} \equiv x_s/x_t$*):*

$$\mathbb{E}_t\left[\xi_T^{\vartheta}\right] = \xi_t^{\vartheta} \mathrm{e}^{-\vartheta\frac{1}{2}\theta'\theta(T-t)} \mathbb{E}_t\left[\mathrm{e}^{-\vartheta\int_t^T r_s ds - \vartheta\theta'(W_T - W_t)}\right]$$

$$\mathbb{E}_t[\xi_T P_T] = \xi_t \mathrm{e}^{-\frac{1}{2}\theta'\theta(T-t)} \int_T^{T_{\max}} \mathrm{e}^{-M(T,s)} \mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r_v dv - \theta'(W_T - W_t) - r_T N(T,s)}\right] ds$$

$$\mathbb{E}_t\left[\int_t^T \xi_s e_s ds\right] = \xi_t e_t \int_t^T \mathrm{e}^{-\frac{1}{2}(\theta'\theta + \sigma_e'\sigma_e)(s-t) + \int_t^s \mu_u^e du} \mathbb{E}_t\left[\mathrm{e}^{-\int_t^s r_u du + (\sigma_e - \theta)'(W_s - W_t)}\right] ds$$

$$\mathbb{E}_t\left[\xi_T H_{t,T}^{P,0}\right] = -R_t \left(\int_T^{T_{\max}} e^{-f_{T,s} - M(T,s)} \mathbb{E}_t^Q\left[R_{t,T} e^{-r_T N(T,s)} \mathcal{D}_t(r_T) N(T,s)\right]\right) ds$$

$$\mathbb{E}_t\left[\xi_T H_{t,T}^{\xi}\right] = R_t \mathbb{E}_t^Q\left[R_{t,T} \int_t^T \mathcal{D}_t(r_s) ds\right]$$

$$\mathbb{E}_t\left[\xi_T^{\vartheta} H_{t,T}^{P,0}\right]$$
$$= -\xi_t^{\vartheta} e^{-\vartheta\frac{1}{2}\theta'\theta(T-t)} \int_T^{T_{\max}} \left(e^{-f_{T,s} - M(T,s)} N(T,s) \mathbb{E}_t\left[e^{-\vartheta\int_t^T r_u du - \vartheta\theta'(W_T - W_t) - r_T N(T,s)} \mathcal{D}_t(r_T)\right]\right) ds$$

$$\mathbb{E}_t\left[\xi_T^{\vartheta} P_T H_{t,T}^{\xi}\right] = \mathbb{E}_t\left[\xi_T^{\vartheta} P_T \left(\int_t^T \mathcal{D}_t(r_s) ds\right)\right]$$

$$\mathbb{E}_t\left[\int_t^T \xi_s e_s H_{t,s}^{\xi} ds\right] = R_t e_t \mathbb{E}_t^Q\left[\int_t^T R_{t,s}(e_{t,s}) \left(\int_t^s \mathcal{D}_t(r_u) du\right) ds\right]$$

where

$R_t \equiv \exp\left(-\int_0^t r_s ds\right)$, and $\mathcal{D}_t(r_s) = -\mathrm{e}^{-\kappa_r(s-t)}\sigma_r$.

Simulations of interest quantities

163. Simply put, the computation of the quantities of interest requires a simulation of a system that involves variables (random): $\int_t^T r_s ds =: \ln(R_{t,T})$, r_T *and* $W_T - W_t =: M_T$ and where the first two depend only on W_B, the third corresponds to the complete vector of Brownian movements; $W_T = (W_B, W_1, \dots, W_{d-1})' \in \mathbb{R}^d$.. The system can be simulated through the following characterization [see, for example, Glasserman (2004, pg. 115)]:

$$r_{t_{i+1}} = e^{-\kappa_r(t_{i+1}-t_i)} r_{t_i} + \mu_r(t_i, t_{i+1}) - \sigma_r(t_i, t_{i+1}) Z_{r,t_{i+1}}$$

$$\ln\left(R_{t_{i+1}}\right) = \ln\left(R_{t_i}\right) + \mu_{R,1}(t_i, t_{i+1}) r_{t_i} + \mu_{R,2}(t_i, t_{i+1}) + \sigma_R(t_i, t_{i+1})\left(\rho_{R,r}(t_i, t_{i+1}) Z_{r,t_{i+1}} + \sqrt{1-\rho_R^2(t_i, t_{i+1})} Z_{R,t_{i+1}}\right)$$

$M_{t_{i+1}}^k = M_{t_i}^k + \sqrt{t_{i+1}-t_i} \times \left(\rho_{M^k,r}(t_i, t_{i+1}) Z_{r,t_{i+1}} + \rho_{M^k,R}(t_i, t_{i+1}) Z_{R,t_{i+1}} + \cdots + \rho_{k,k-1}(t_i, t_{i+1}) Z_{k-1,t} + \sqrt{1-\rho_k^2(t_i, t_{i+1})} Z_{k,t_{i+1}}\right)$where $M_{t_{i+1}}^k \sim N(0, (t_{i+1}-t_i))$, $r_{t_{i+1}} \sim N(e^{-\kappa_r(t_{i+1}-t_i)} r_{t_i} + \mu_r(t_i, t_{i+1}), \sigma_r^2(t_i, t_{i+1}))$, $\ln R_{t_{i+1}} \sim N(\ln R_{t_i} + \mu_R(t_i, t_{i+1}), \sigma_R^2(t_i, t_{i+1}))$, $\left\{\rho_{k,k-1}\right\}_{k\in\{2,\dots,d-1\}}$,while ρ_R^2, $\rho_{r,R}$, $\rho_{M,r}$, $\rho_{M,R}$, $\rho_{k,k-1}$ y ρ_k^2 are analogs of the described quantities at the beginning of this section for the financial asset base.

171. In addition, the specific coefficients are given:

$$\mu_r(t_i, t_{i+1}) = \bar{r}\left(1 - e^{-\kappa_r(t_{i+1}-t_i)}\right)$$

$$\sigma_r(t_i, t_{i+1}) = -\sigma_r\sqrt{\frac{\left(1 - e^{-2\kappa_r(t_{i+1}-t_i)}\right)}{2\kappa_r}}$$

$$\mu_{R,1}(t_i, t_{i+1}) = \frac{1}{\kappa_r}\left(1 - e^{-\kappa_r(t_{i+1}-t_i)}\right)$$

$$\mu_{R,2}(t_i, t_{i+1}) = \kappa_r\left[(t_{i+1}-t_i) - \mu_{R,1}(t_i, t_{i+1})\right]$$

$$\sigma_R(t_i, t_{i+1}) = -\frac{\sigma_r}{\kappa_r}\left((t_{i+1}-t_i) + \frac{1}{2\kappa_r}\left(1 - e^{-2\kappa_r(t_{i+1}-t_i)}\right) - \frac{2}{\kappa_r}\left(1 - e^{-\kappa_r(t_{i+1}-t_i)}\right)\right)^{1/2}$$

$$\rho_{R,r}(t_i, t_{i+1}) = \frac{\sigma_{R,r}(t_i, t_{i+1})}{\sigma_r(t_i, t_{i+1}) \cdot \sigma_R(t_i, t_{i+1})}$$

$$\sigma_{R,r}(t_i, t_{i+1}) = \frac{\sigma_r^2}{2\kappa_r}\left[1 + e^{-2\kappa_r(t_{i+1}-t_i)} - 2e^{-\kappa_r(t_{i+1}-t_i)}\right]$$

$$\rho_R^2(t_i, t_{i+1}) = \rho_{R,r}^2(t_i, t_{i+1})$$

$$\rho_{M^k,r}(t_i, t_{i+1}) = -\frac{\sigma_{M^k,r}}{\sigma_r\sqrt{t_{i+1}-t_i}}$$

$$\sigma_{M^k,r}(t_i, t_{i+1}) = -\frac{\sigma_r}{\kappa_r}\left(1 - e^{-\kappa_r(t_{i+1}-t_i)}\right)$$

$$\rho_{M^k,R}(t_i, t_{i+1}) = -\frac{\sigma_{M^k,R}}{\sigma_r\sqrt{t_{i+1}-t_i}}$$

$$\sigma_{M^k,R}(t_i, t_{i+1}) = -\frac{\sigma_r}{\kappa_r}\left(t_{i+1} - t_i + \frac{1}{\kappa_r} e^{-\kappa_r(t_{i+1}-t_i)}\right)$$

$$\rho_{k,k-1}(t_i, t_{i+1}) = \frac{\sigma_{M^k,M^{k-1}}}{t_{i+1} - t_i}$$

164. Likewise, the conditional expectations can be approximated as the following:

$$\mathbb{E}_t\left[\mathrm{e}^{-\vartheta\int_t^T r_s ds - \vartheta\theta'(W_T - W_t)}\right] \cong \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{-\vartheta\left(\ln\left(R_{t_N}^{(n)}\right) - \ln\left(R_{t_{N-1}}^{(n)}\right)\right) - \vartheta\theta'\left(M_{t_N}^{(n)} - M_{t_{N-1}}^{(n)}\right)},$$

$$\mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r_v dv - \theta'(W_T - W_t) - r_T N(T,s)}\right] \cong \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{-\left(\ln\left(R_{t_N}^{(n)}\right) - \ln\left(R_{t_{N-1}}^{(n)}\right)\right) - \theta'\left(M_{t_N}^{(n)} - M_{t_{N-1}}^{(n)}\right) - r_{t_N}^{(n)} N(t_N,s)},$$

$$\mathbb{E}_t\left[\mathrm{e}^{-\int_t^s r_u du + (\sigma_e - \theta)'(W_s - W_t)}\right] \cong \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{-\left(\ln\left(R_{t_N}^{(n)}\right) - \ln\left(R_{t_{N-1}}^{(n)}\right)\right) + (\sigma_e - \theta)'\left(M_{t_N}^{(n)} - M_{t_{N-1}}^{(n)}\right)},$$

$$\mathbb{E}_t\left[e^{-\vartheta\int_t^T r_u du - \vartheta\theta'(W_T - W_t) - r_T N(T,s)} \mathcal{D}_t(r_T)\right] \cong -\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{-\vartheta\left(\ln\left(R_{t_N}^{(n)}\right) - \ln\left(R_{t_{N-1}}^{(n)}\right)\right) - \vartheta\theta'\left(M_{t_N}^{(n)} - M_{t_{N-1}}^{(n)}\right) - r_{t_N}^{(n)} N(t_N,s) - \kappa_r(t_N - t_{N-1})} \sigma_r,$$

$$\mathbb{E}_t\left[\int_t^T \xi_{t,s}(e_{t,s})\left(\int_t^s \mathcal{D}_t(r_u) du\right) ds\right] \cong -\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{-\left(\ln\left(R_{t_N}^{(n)}\right) - \ln\left(R_{t_{N-1}}^{(n)}\right)\right) - \theta'\left(M_{t_N}^{(n)} - M_{t_{N-1}}^{(n)}\right) - r_{t_N}^{(n)} N(t_N,s) - \kappa_r(t_N - t_{N-1})} \sigma_r,$$

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